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Similar Solutions of the Boundary Layer Equation for Power Law Fluids

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Abstract:
The present paper provides the boundary layer equation for the two-dimensional flow of a power law fluid along with solutions of free stream velocity u(x) and scaling function g(x). We have also discussed here the theory of similar solutions of the boundary layer equations for power low fluids on the same lines as is usually done for Newtonian fluids.

Key Words : Power law, boundary layer, fluid, similar solutions, Newtonian fluid.

1. INTRODUCTION

A power low fluid is heated by passing it under conditions of laminar flow through a long tube whose wall temperature varies in the direction of flow [1]. Fluid flows through porous media present important applications in many fields of engineering and science. For instance, it is very relevant in geometrics due to the distinct rock and soil types varying properties where sub surface flows occur [2, 5-8, 11]. Here we discuss the theory for similar solutions of the boundary layer equations for power law fluids on the same lines as is usually done for Newtonian fluids. In this manner we obtain a generalisation of the Falker-Skan equation.

2. BASIC EQUATIONS

The boundary layer equation for the two-dimensional flow of a power-law fluid are, as obtained by Kapur [3, 4]

$$u\frac{\delta u}{\delta x}+v\frac{\delta u}{\delta y}=u\frac{du}{dx}+\frac{v\delta}{\delta y}\left[\left|\frac{\delta u}{\delta y}\right|^{n-1}.\frac{\delta u}{\delta y}\right] \qquad (2.1)$$

and $\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}=0$ (2.2)

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where u and v are two components of velocity along and perpendicular to the wall. U(x) is the free stream velocity, ρ is the density of the fluid and

$$\gamma = 2^{\frac{n-1}{2}}.\mu/\rho$$

The equation of continuity is integrated by introducing the stream function $\psi(x,y)$ which is such that

$$u = \frac{\delta\psi}{\delta y}; v = -\frac{\delta\psi}{\delta x} \tag{2.3}$$

The boundary layer equations then reduce to

$$\frac{\delta\psi}{\delta y}.\frac{\delta^2\psi}{\delta x\delta y} - \frac{\delta\psi}{\delta x}.\frac{\delta^2\psi}{\delta y^2} = U(x)\frac{dU(x)}{dx} + \gamma\frac{\delta}{\delta y}\left[\left|\frac{\delta^2\psi}{\delta y^2}\right|^{n-1}.\frac{\delta^2\psi}{\delta y^2}\right] \tag{2.4}$$

We introduce new variables and functions;

$$\xi = \frac{x}{L}; \eta = \frac{y.R_1^{\frac{1}{n+1}}}{Lg(x)} \tag{2.5}$$

$$f(\xi,\eta) = \frac{\psi(x.y)R_1^{\frac{1}{n+1}}}{L.U(x).g(x)} \tag{2.6}$$

Where R_1 is the modified Reynold's number defined by

$$R_1 = \frac{L^n}{\gamma U_\infty^{h-2}} \tag{2.7}$$

L and U_∞ are the reference length and velocity respectively, and g(x) is a suitable scaling function to be chosen later. In terms of these variables, we get

$$u = \frac{\delta\psi}{\delta y} = U(x)\frac{\delta f}{\delta\eta} \tag{2.8}$$

$$v = -\frac{\delta\psi}{\delta x} = -\left[L.f\frac{d}{dx}(Ug) + Ug\left(\frac{\delta f}{\delta\xi} - \frac{g^1}{g}L\eta\frac{\delta\xi}{\delta\eta}\right)\right]/R^{\frac{1}{n+1}} \tag{2.9}$$

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$$\frac{\delta u}{\delta x} = U^1(x)\frac{\delta f}{\delta \eta} + \frac{U(x)}{L}\left[\frac{\delta^2 f}{\delta\xi\delta\eta} - L\eta\frac{\delta^2 f}{\delta\eta^2}.\frac{g^1(x)}{g(x)}\right] \tag{2.10}$$

$$\frac{\delta u}{\delta y} = U(x).\frac{\delta^2 f}{\delta\eta^2}.\frac{\eta}{y} \tag{2.11}$$

Substituting in (2.1) and remembering that for similar solutions f(x, y) should be independent of ξ, we get after consideration simplifications, the basic equation as

$$| f^n |^{n-1} .f''' + \alpha ff'' + \beta(1 - f^{1^2}) = 0 \tag{2.12}$$

where

$$\alpha = \frac{Lg^n_{(x)}U^{n-2}_\infty}{nU^{n-1}_{(x)}}\frac{d}{dx}(U_{(x)}.g_{(x)}) \tag{2.13}$$

and

$$\beta = \frac{Lg^{n+1}_{(x)}U^{n-2}_\infty}{nU^{n-1}_{(x)}}U^1_{(x)} \tag{2.14}$$

We choose U(x) and g(x) so that α and β are constants. For n = 1, the equation (2.12) reduces to the well known Falker-Skan equations.

$$f^m + \alpha\ ff'' + \beta(1 - f^{1^2}) = 0 \tag{2.15}$$

with

$$\alpha = \frac{Lg(x)}{U_\infty}\frac{d}{dx}(U_{(x)}.g_{(x)})$$

$$\beta = \frac{Lg^2_{(x)}U^1_{(x)}}{U_\infty} \tag{2.16}$$

In general the velocity component u will increase from its zero value at the wall to the value U(x) at the edge of the boundary layer and thus in this case $\delta u/\delta y$ would be non-negative. From (3.6) and (2.11) it would then appear that if g(x) can be chosen to be a non-negative

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function, we can take f'' to be non-negative. Thus we can make the assumption that f'' is non-negative function and integrate.

$$f''^{n-1}.f''' + \alpha\, ff' + \beta(1 - f'^{2}) = 0 \qquad (2.17)$$

Subject to the boundary conditions.

$\eta = 0$ for f = 0 and $f^1 = 0$

$$\eta = \infty \text{ for } f^1 = 1 \qquad (2.18)$$

The assumption made can then be tested against the solution so obtained.

3. SOLUTION FOR U(x) AND g(x)

From (2.13) and (2.14) we have

$$\frac{d}{dx}\left[g_{(x)}^{h+1}.U_{(x)}^{2-n}\right] = \frac{n}{LU_{\infty}^{h-2}}[(n+1)\alpha - (2n-1)\beta] \qquad (3.1)$$

Integrating the above equation for

$$(n+1)\alpha - (2n-1)\beta \neq 0$$

We get

$$g_{(x)}^{h+1}.U_{(x)}^{2-n} = \frac{n}{LU_{\infty}^{n}}\left[(n+1)\alpha - (2n-1)\beta\right]x \qquad (3.2)$$

Also from (5.2.13) and (5.2.14) we get

$$\frac{U'(x)}{U(x)}(\alpha - \beta) = \beta\frac{g'(x)}{g(x)} \qquad (3.3)$$

which gives on integration

$$\left(\frac{U_{(x)}}{U_\infty}\right)^{\alpha-\beta} = kg_{(x)}^{\beta} \tag{3.4}$$

where k is a dimensionless constant.

Solving for U(x) and g(x) we get

$$\left(\frac{U_{(x)}}{U_\infty}\right)^{\frac{\alpha-\beta}{\beta}+\frac{2-n}{n+1}} = K^{\frac{1}{\beta}}\left[\{(n+1)\alpha-(2n-1)\beta\}\frac{nx}{L}\right]^{\frac{1}{n+1}} \tag{3.5}$$

and

$$\left[g_{(x)}\right]^{1+\frac{\beta(2-n)}{(\alpha-\beta)(n+1)}} = k^{-\frac{2-n}{(n+1)(\alpha-\beta)}}\left[\{(n+1)\alpha-(2n-1)\}\beta\frac{nx}{L}\right]^{\frac{1}{n+1}} \tag{3.6}$$

From (2.13) and (2.14) it is seen that the result is independent of any common factor of α and β and, as it can be includd in g(x). Therefore as long as $a \neq 0$, we can put $\alpha = 1$ without loss of generally. Also introducing a new parameter m defined by

$$m = \frac{\beta}{(n+1)\alpha-(2n-1)\beta} \text{ or } \beta = \frac{m(n+1)}{1+m(2n-1)} \tag{3.7}$$

we get

$$\left(\frac{U_{(x)}}{U_\infty}\right) = (k)^{1+m(2n-1)} = \left(\frac{n(n+1)}{1+m(2n-1)}.\frac{x}{L}\right)^{m} \tag{3.8}$$

or simply $U_{(x)} = C.x^{m}$

Where C is a constant of propertionslity.

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and $g(x)=\left[\frac{n(n+1)}{1+m(2n-1)}.\frac{z}{L}.\left(\frac{U_\infty}{U_{(x)}}\right)^{2-n}\right]^{\frac{1}{n+1}}$ (3.9)

or simply $g(x)=C_1(z)^{\frac{1-2m+mn}{n+1}}$

where C_1 is a constant of proportionslity.

Also from (2.8) and (3.9)

$$\eta=\left[\frac{1+m(2n-1)}{n(n+1)}-\frac{U_\infty^{2-n}}{x^v}\right]^{\frac{1}{n+1}} \tag{3.10}$$

The case $(n+1)\alpha-(2n-1)\beta=0$ left earlier leads to the

$$g_{(x)}^{n+1}.U_{(x)}^{2-n}=\text{constant}=C \tag{3.11}$$

$$(n+1).g_x^n.g^1(x)=C(n-2)U_x^{n-3}U^1(x)$$

and $(\alpha-\beta)=C.L.U_\infty^{n-2}.\frac{(n-2)}{n(n+1)}.\frac{U^1(x)}{U(x)}$ (3.12)

Integration of (3.12) gives

$$U(x)=C_2e^{\left(\frac{1-2m+mn}{n+1}\right)x} \tag{3.13}$$

and also $g(x)=C_3e\frac{(n-2)(1-2m+mn)}{(n+1)^2}.x$ (3.14)

where C_2 and C_3 are constants of proportionality.

4. SCHUWALTER DISCUSSION OF SIMILARITY SOLUTIONS

Schuwalter (1960) [9, 10] has independently attacked the problems of similarity solutions in three-dimensional case of the flow past a flat plate where potential velocity vector is not perpendicular to the leading edge, and this is a much more restrictive result than is obtained for Newtonian fluids in three dimensional flow.

The object of the present section is to point out a discrepancy is the mathematical development of Schuwalter (1960) and to give necessary corrections.

We use the notations or Schowalter's paper. Equation (14), (15) and (16) of the text are

$$\frac{g^{n+}}{(U^0)^{n-1}}\cdot\frac{\delta U^0}{\delta x^0}\cdot\left[F'^2 - FF'' - 1\right] + \frac{g^{n+1}.W^0\delta U^0}{U^0}\frac{\delta U^0}{\delta z^0}\left[F'G' - 1\right]$$

$$-\frac{g^{n+1}}{(U^0)^{n-1}}\frac{\delta w^0}{\delta z^0}GF^0 - \frac{g^n}{(U^0)^{n-2}}\frac{\delta g}{\delta x^0}FF'' - \frac{g^n w^0}{(U^0)^{n-1}}\frac{\delta g}{\delta x^0}F''G$$

$$= R^{N(n+1)-1}.\frac{\delta}{\delta\eta}\left\{\left[F''^2 + \left(\frac{W}{U}G''\right)\right]^{\frac{n-1}{2}}F''\right\} + \frac{KLg^{n+1}}{U_\infty^2(U^0)^n}f_{2x};$$

$$\frac{1}{(U^0)^{n-1}}\cdot\frac{\delta U^0}{\delta x^0}\cdot\left[F'^2 - FF''-1\right] + W^0\frac{\delta tuU^0}{\delta x^0}\left[F'G'-1\right]$$

$$-\frac{1}{(u^0)^{n-1}}\cdot\frac{\delta w^0}{\sigma x^0}GF'' - \frac{1}{(U^0)^{n-1}}\frac{\delta lug}{\delta x^0}FF'' - \frac{W^0}{(U^0)^{n-1}}\frac{\delta lug}{\delta x^0}F''G$$

$$= \frac{1}{g^{n+1}}\frac{\delta}{\delta n}\left\{\left[F''^2 + \left(\frac{W}{U}G''\right)^2\right]^{\frac{n-1}{2}}.F\right\} + \frac{KL}{U_\infty^2(U^0)^n}.f_{2x};$$

$$\frac{1}{(W^0)^{n-1}}\cdot\frac{\delta w^0}{\delta z^0}\left[G'^2 - GG''-1\right] + U^0\frac{\delta\, In\, w^0}{\delta x^0}\left[F'G'-1\right]$$

$$-\frac{1}{(w^0)^{n-1}}\frac{\delta U^0}{\delta x^0}.FG'' - \frac{1}{(W^0)^{n-2}}\frac{\delta lug}{\delta z^0}GG'' - \frac{U^0}{(W^0)^{n-1}}\cdot\frac{\delta l_u}{\delta x^0}FG''$$

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$$= \frac{1}{g^{n+1}} \cdot \frac{\delta}{\delta\eta} \left\{ \left[G'^2 + \left(\frac{U}{W} F'' \right)^2 \right]^{\frac{h-1}{2}} G' \right\} + \frac{KL}{U_\infty^2 (W^0)^n} \cdot f_{2x};$$

Their correct forms should be respectively.

$$\frac{g^{n+1}}{(U^0)^{n-1}} \frac{\delta U^0}{\delta x^0} \left[F'^2 - FF'' - 1 \right] + \frac{g^{n+1} W^0}{(U^0)^n} \cdot \frac{\delta U^0}{\delta z^0} \left[F'6 - 1 \right]$$

$$- \frac{g^{n+1}}{(U^0)^{n-1}}, \frac{\delta W^0}{\delta x^0} .GF'' - \frac{g^n}{(U^0)^{n-2}} \frac{\delta g}{\delta x^0} FF'' - \frac{g^n W^0}{(U^0)^{n-2}} \frac{\delta g}{\delta z^0} FF''$$

$$= (R)^{N(n+1)-1} \frac{\delta}{\delta\eta} \left\{ \left[F''^2 + \left(\frac{W}{u} G'' \right)^2 \right]^{\frac{n-1}{2}} F'' \right\} + \frac{K.Lg^{n+1}}{U_\infty^2 (U^0)^n} f_{2x}; \tag{4.1}$$

$$\frac{1}{(U^0)^{n-1}} \frac{\delta U^0}{\delta x^0} \left[F'^2 - FF'' - 1 \right] + \frac{W^0}{(U^0)^{n-1}} \cdot \frac{\delta u U^0}{\delta z^0} [F'G' - 1]$$

$$- \frac{1}{(U^0)^{n-1}} \cdot \frac{\delta W^0}{\delta z^0} .GF'' - \frac{1}{(U^0)^{n-1}} \frac{\delta lug}{\delta x^2} F''F - \frac{W^0}{(v^0)^{n-1}} \frac{\delta luj}{\delta x^0} F''G$$

$$= \frac{1}{g^{n+1}} \cdot \frac{\delta}{\delta\eta} \cdot \left\{ \left[F''^2 + \left(\frac{W}{U} G'' \right)^2 \right]^{\frac{n-1}{2}} F'' \right\} + \frac{K.L}{U_\infty^2 (U'')^n} f_{2y} \tag{4.2}$$

and

$$\frac{1}{(W^0)^{n-1}} \cdot \frac{\delta W^0}{\delta x^0}, \left[G'^2 - GG'' - 1 \right] + \frac{U^0}{(W^0)^{n-1}} \cdot \frac{\delta lu W^0}{\delta x^0} [F'G' - 1]$$

$$- \frac{1}{(W^0)^{n-1}} \cdot \frac{\delta U^0}{\delta x^0}, FG'' - \frac{1}{(W^0)^{n-2}} \cdot \frac{\delta lug}{\delta x^0} GG'' - \frac{U^0}{(W^0)^{n-1}} \frac{\delta lug}{\delta x^0} FG''$$

$$= \frac{1}{g^{n+}} \cdot \frac{\delta}{\delta\eta} \left\{ \left[G''^2 + \left(\frac{U}{W} F'' \right)^2 \right]^{\frac{n-1}{2}} G'' \right\} + \frac{K.L}{U_\infty^2 (W^0)^n} . f\, 2z \tag{4.3}$$

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Thus the modified form of the equation (17) of the above text (Schuwalter paper) is

$$(U^0)^{1-n}\frac{\delta U^0}{\delta x^2}=a_1 W^0(U^0)^{1-n}\frac{\delta l_u U^0}{\delta x^2}=\alpha_2(U^0)^{1-n}\frac{\delta w^0}{\delta z^0}$$

$$=a_3(U^0)^{2-n}\frac{\delta ug}{\delta x^0}=a_n(v^0)^{1-n}W^0\frac{\delta ug}{\delta x^0}$$

$$=a_5\frac{1}{g^{n+1}}=a_6(W^0)^{1-n}.\frac{\delta W^0}{\delta x^0}$$

$$=a_7 U^0(W^0)^{1-n}.\frac{\delta W^0}{\delta x^0}=a_8(W^0)^{1-n}.\frac{\delta U^0}{\delta x^0}$$

$$=a_9(W^0)^{2-n}.\frac{\delta lug}{\delta x^0}=a_{10}(W^0)^{1-n}.\frac{\delta \mu g}{\delta x^0} \qquad (4.4)$$

The requirement of proportionality between U^0 and W^0 as obtained from (4.4) may be stated as

$$W^0=k_{10}U^0 \qquad (4.5)$$

and therefore (4.4)

$$(U^0)^{1-n}\frac{\delta U^0}{\delta x^0}=a_1.K_0(U^0)^{1-n}\frac{\delta U^0}{\delta x^0}=a_2 k_0(U^0)^{1-n}\frac{\delta U^0}{\delta x^2}$$

$$=a_3\frac{(U^0)^{2-n}}{g}\frac{\delta y}{\delta x^2}=a_4 k_n\frac{(U^0)^{2-n}}{g}.\frac{\delta g}{\delta x^0}$$

$$=a_5\frac{1}{g^{n+1}}=a_6(v_0U^0)^{1-n}K_0\frac{\delta U^0}{\delta z^0}$$

$$=a_7(k_0U^0)^{1-n}.\frac{\delta U^0}{\delta x^2}=a_8(K_0U^0)^{1-n}.\frac{\delta U^0}{\delta x^2}$$

$$=a_9\frac{(k_0U^0)^{2-n}}{g}.\frac{\delta g}{\delta x^2}=a_{10}K_0^{1-n}\frac{(Ug)^{2-h}}{g}\frac{\delta g}{\delta x^0} \qquad (4.6)$$

where K_0 is a constant, and

$$1=a_7K_0^{1-n}=U_8K_0^{1-n};a_1=a_2=a_6k_0^{1-n}$$

$$a_2=a_{10}K_0^{1-n}\ :\ a_4=a_gk_0^{1-n} \qquad (4.7)$$

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we also get

$$\frac{\delta \ln U^0}{\delta x^0} = a_1.K_0.\frac{\delta \ln g}{\delta z^0} = a_3 \frac{\delta \ln g}{\delta z^0}$$

$$= a_4 k_0 \frac{\delta \ln g}{\delta z^0} = a_2 \frac{(U^0)^{n-2}}{g^{n+1}} \tag{4.8}$$

Integrating (4.8) for U^0 and g separately, we get

$$U^0 = T(x^0 + Az^0) \tag{4.9}$$

$$g = [f(x^0 + Az^0)]^p \tag{4.10}$$

Where

$$1 = a, k_0 A = a_3 p = a_4 k_a.A.p \tag{4.11}$$

and

$$f' = a_5 f^{[(n-1)-p(n+1)]} \tag{4.12}$$

Here A and p are arbitrary constants and f is an arbitrary function. The constant p corresponds to $\frac{\alpha - 3}{\beta}$ of our discussions in the proceeding sections.

Integration (4.12) and using (4.5), we get

$$U^0 = C(X^0 + Az^0)^m \tag{4.13}$$

$$W^0 = ck_0(x^n + Az^0)^m \tag{4.14}$$

and

$$g = c^p (x^0 + Az^0)^{pm} \tag{4.15}$$

where $m = \frac{1}{[p(n+1)-(n-2)]}$

top the case when $p(n+1)-(n-2) \neq 0 \ and \ a \neq 0$.

If p (n + 1) – (n – 2) = 0 ; we get the solutions

$$U^0 = Ce^{B.p(x^0+Az^0)} \tag{4.16}$$

$$W^0 = k_0^{\ Bp(x^0+Az^0)} \tag{4.17}$$

$$g = C'e^{B(x^0 + Az^0)p\frac{n-2}{n+1}} \tag{4.18}$$

In the above equation, we can regard k_0, B, C and C' as arbitrary constants and then (4.17) and (4.18) would determine a_1 to a_{10}.

From (4.18), we find that the potential velocity is in a fixed direction and from (4.13) and (4.14) we find that "similar solution" are possible when the potential velocity is proportional to some power of the distance from a fixed stagnation straight line.

5. DISCUSSION

By putting $k_0 = 0$ and A = 0, we get the two dimensional case and the corresponding equations for

$P(n+1) - (n-2) \neq 0 \;\; and\, a = 1$ reduce to

$$U^0(x^0) = c(x^0)^m \tag{5.1}$$

which is the same as (3.8)

and

$$g(x^0) = c^p (x^0)^{\frac{1-2m+mn}{n+1}}$$

Which again is the case as (3.9)

For p(n + 1) – (n – 2) = 0 equation (4.16) reduces to

$$U^0(x^0) = Ce^{Bpx^0}$$

$$= c_1 e\left(\frac{1-2m+mn}{n+1}\right)x^0 \tag{4.2}$$

Which again is the same as the value of U(x) obtained in equation (3.13), also we get

$$g(x^0) = C'e^{Bp\frac{n-2}{n+1}x^0}$$

$$= C'e\frac{(n-2)(1-2m+mn)}{(n+1)^2}x^0 \tag{4.3}$$

which is the same as (3.14). Thus we see that Schualter's statement gives result which are the same as ours after the mathematical error is corrected.

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