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FINDING LYAPUNOV EXPONENTS AND LOOKING AT TIMESERIES GRAPHS ON NONLINEAR CHAOTIC SYSTEMS

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1.1 Abstract

In this paper, we use Lyapunov exponents to confirm the chaotic region and look at the graphs of time series analysis to back up our periodic orbits of periods 2^0, 2^1, 2^2 and so on, as well as the chaotic behavior on nonlinear discrete model:

$$\mu: [0,4] \rightarrow [0,4], \quad \mu(x) = cx^2 - dx$$

in which c = -1 and d is a tunable parameter in the range of [-4, -1].

Keywords:

1.2 Introduction

It has long been known that many nonlinear dynamical systems responses swirl around in a random like, seemingly irregular, but clearly defined manner rather than following straightfo ward, regular, and predictable paths. Such complex behavior can arise from even a simple strictly deterministic model, provided that the process involved is non-linear.

Lyapunov exponents, or rates of orbital divergence or convergence, are unquestionably crucial to the understanding of chaos. Orbital divergence and chaos are indicated by positive Lyapunov exponents, which also establish the time scale at which state prediction is not feasible. The time scale on which transients or perturbations of the system's state will fade is determined by negative Lyapunov exponents [3, 4, 5, 7].

The idea of a time series of data for the system is the primary theoretical instrument for characterizing chaotic behavior [5]. One of the intriguing characteristics of chaotic systems is orbit complexity. The term "orbit complexity" refers to the unlimited number of unstable periodic orbits that coexist with the weird attractor and are crucial to the dynamics of the system in chaotic systems. However, in many real-world scenarios, one must deal directly with experimental data and lacks access to system equations the way a time series looks.

In this paper, we calculate Lyapunov exponents to validate the chaotic region and present time series analysis graphs to bolster our periodic orbits for periods 2^0, 2^1, 2^2,… in relation to the nonlinear discrete model:

$$\mu: [0,4] \rightarrow [0,4], \quad \mu(x) = cx^2 - dx$$

in which c = -1 and d is a tunable parameter in the range of [-4, -1].

1.2 Calculating Lyapunov Exponents

In order to formally calculate Lyaponuv exponents, we start by taking into account an attracto r point x_0 and a nearby attractor point $x_0 + \varepsilon$.

Next, for every value, we apply the iterated map function μ, n times while taking into

$$E_n = \left|\mu^n(x_0+\varepsilon) - \mu^n(x_{0)}\right|$$

If the behavior is chaotic, we anticipate that the separation above will increase exponentially with n. Therefore, we may write

$$E_n = \varepsilon e^{\gamma n} \implies \gamma = \frac{1}{n}\left[\frac{\left|\mu^n(x_0+\varepsilon)-\mu^n(x_{0)}\right|}{\varepsilon}\right]$$

where γ is the trajectory's Lyaponuv exponent.
By using the chain rule for differentiation and allowing ε to fall to zero, γ can be expressed in more understandable way in the form:

$$\gamma(the\ rate\ of\ divergence\ of\ the\ two\ trajectories) = \frac{1}{n}\log(\left|\mu^/(x_0)\right|\left|\mu^/(x_1)\right| \ldots. \left|\mu^/(x_{n-1}\right|)$$

where $\mu^n(x_0) = \mu^/(x_0)\mu^/(x_1)\ldots\mu^/(x_{n-1)}$

This implies $\gamma = \frac{1}{n}\sum_{i=0}^{n-1} log\left|\mu^/(x_i)\right|$

The beginning value may have an impact on the Lyapunov exponent's value. Therefore, by taking an appropriate number of points at a time, we may calculate the average Lyapunov exponent. To obtain the Lyapunov exponent, we follow the above steps with the aid of a computer program.

For a particular value of the parameter d, if $x_1^s, x_2^s, \ldots\ldots..x_n^s$ are n stable periodic points, then the Lyapunov exponent

$$\gamma = \frac{1}{n}\sum_{i=0}^{n-1} log\left|\mu^/(x_i)\right| \text{ , (where } x_0 \text{ is the initial value)}$$

becomes

$$\gamma = \sum_{i=1}^{n} log\left|\mu^/(x_i^s)\right|$$

When x_0 is sufficiently close to one of x_i^s, i=1,2, 3,…..n.

This implies $\gamma < 0$, for

$$\left|\mu^/(x_1^s)\mu^/(x_2^s)\ldots..\mu^/(x_n^s)\right| < 1$$

Therefore, as long as there are stable periodic points, γ will be negative. Nevertheless, if the normal maxima is a component of the attractor for that specific value of the parameter d, γ =-∞, as

$|\mu^/(x_i^s)|=0$ for some value of i where the value of μ is at its maximum.

Let's say

$\hat{d}$ be the bifurcation value and $\hat{d}-\delta$ be a parameter value where 2^n stable periodic point occurs. Then as $\delta \to 0^+$,

$$\mu^1(x_1^s).\mu^1(x_2^s)\ldots\ldots\ldots..\mu^1(x_r^s) = -1+\vartheta$$

where $\vartheta \to 0^+, x_1^s, x_2^s, \ldots..x_r^s, r = 2^n$,
are 2^n stable periodic points.
Hence,

$$\gamma = log|\mu^1(x_1^s).\mu^1(x_2^s)\ldots\ldots\mu^1(x_r^s)| = \lim_{\vartheta\to 0} log|-1+\vartheta| = 0$$

Likewise, for $\hat{d}+\delta$ where 2^{n+1} a stable periodic point occurs, we get $\gamma = 0$

Hence, $\lim_{d\to\hat{d}}(\gamma) = 0, \hat{d}$
is a bifurcation value.

The estimated Lyapunov exponent values for a few parameter variables are provided below, t aking into account an iteration size of 50,000 to determine which values have the maximum v alue.

Table 1.1

Parameter Values	Lyapunov Exponents	Parameter Values	Lyapunov Exponents
d_1	-0.00020001	-3.56993	-0.00504842
d_2	-0.00020242	**-3.56995**	**0.00314049**
d_3	-0.00005442	-3.56994	0.00304046
d_4	-0.00005672	-3.56997	0.00583024
d_5	-0.00018182	-3.57000	0.0073192
d_6	-0.00014146	-3.57200	0.0109608
d_7	-0.00006972	-3.57400	0.0702856
d_8	-0.00006810	-3.57600	0.0881221

The graph of Lyapunov exponents against parameter values ranging from -2.9 to -4.0 is shown below. This figure's primary value is in its ability to clearly differentiate between regions that gravitate to a fixed point or a periodic orbit (i.e., γ < 0) and those that are chaotic (Lyapunov exponent γ > 0). We observe multiple places, the first of which is at d = -3, where the Lyapunov exponent strikes the horizontal line before turning negative once more. These are the bifurcations that double periods. The first three bifurcation points—-3.0, -3.44948974278…, and -3.54409035955—are supported by the figure.
The first column of Table 1.1 displays the Lyapunov exponents computed at the first eight bif urcation points, i.e d_k (k = 1, 2,….,8), where the Lyapunov exponent is nearly zero.
When the parameter value d= -3.56995 (approx.) is reached, the first chaotic region appears. Additionally, following the initial chaotic area, we see that some areas of the graph are in the x-axis's negative side. They represent that axis within the chaotic area as well they indicate that regular behaviors occur within the chaotic region as well at specific parameter values, followed by regular behaviors at additional parameter values, and then the chaotic region begins once more at those same parameter values. These windows actually mark the beginning of the chaotic region. These are the windows in the hectic area, actually.

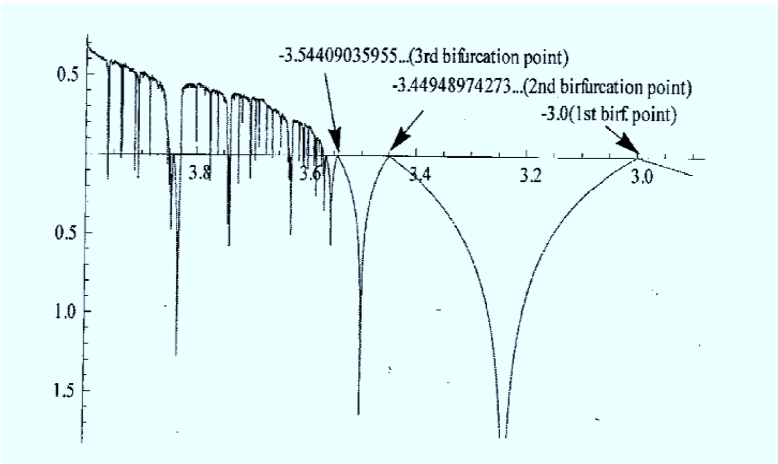


Figure 1.1: Graph of Lyapunov Exponents Versus Parameter d for $-4.0 \leq d \leq -2.9$

Cascades of Period-doubling:

Table 1.2

Period	One of the Periodic Points	Bifurcation Points
1	$x_1 = 2.000000000000 ....$	$d_1 = -3.000000000000 ...$
2	$x_2 = 1.517638080205 ....$	$d_2 = -3.449489742782 ...$
4	$x_3 = 2.905392825135 ....$	$d_3 = -3.544090359553 ...$
8	$x_4 = 3.138826940654 ....$	$d_4 = -3.564407266094 ...$
16	$x_5 = 1.24176887630 ....$	$d_5 = -3.568759419543 ...$
32	$x_6 = 3.178136192507 ....$	$d_6 = -3.569691609802 ...$
64	$x_7 = 3.178152098563 ....$	$d_7 = -3.569891259377 ...$
128	$x_8 = 3.178158223215 ....$	$d_8 = -3.569943176047 ...$
256	$x_9 = 3.178160120724 ....$	$d_9 - 3.569943176047 ...$
512	$x_{10} = 1.696110052279 ....$	$d_{10} = -3.569945137343 ...$
1024	$x_{11} = 1.696240778303 ....$	$d_{11} = -3.569945557392$
…..	….. …. …..	…. …. ….

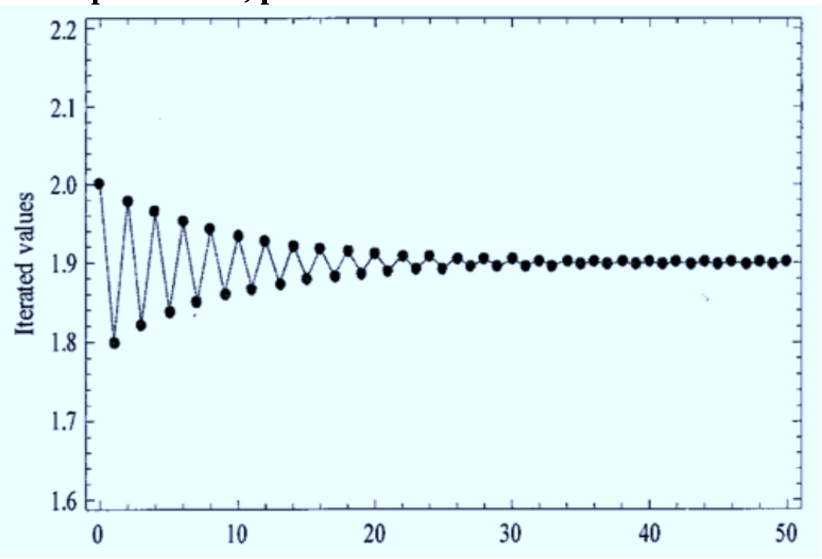
1.2 Analysis of Time Series

The difference equation in our situation is

$$x_{n+1} = cx_n^2 - dx_n, c = -1 \; and \; n = 0,1,2, \dots ....$$

The vertical axis shows the amplitude for each iteration, while the horizontal axis shows the number of iterations (or "time"). Time series analysis graphs are displayed to demonstrate the presence of several periodic orbits with periods 2^k, k = 0, 1, 2, …, as well as chaotic behavior. An attractor is the set that the n values converge towards. As we have shown, an attractor might be a limit cycle, a fixed point, or an attractor that is chaotic. The following figures show the time series graphs.

Initial point x=2.0, parameter value *d* =-2.9



No. of iterations

Figure 1.2: The time Series Showing Behavior of Period One

Initial point x=2.0, parameter value *d* =-3.1

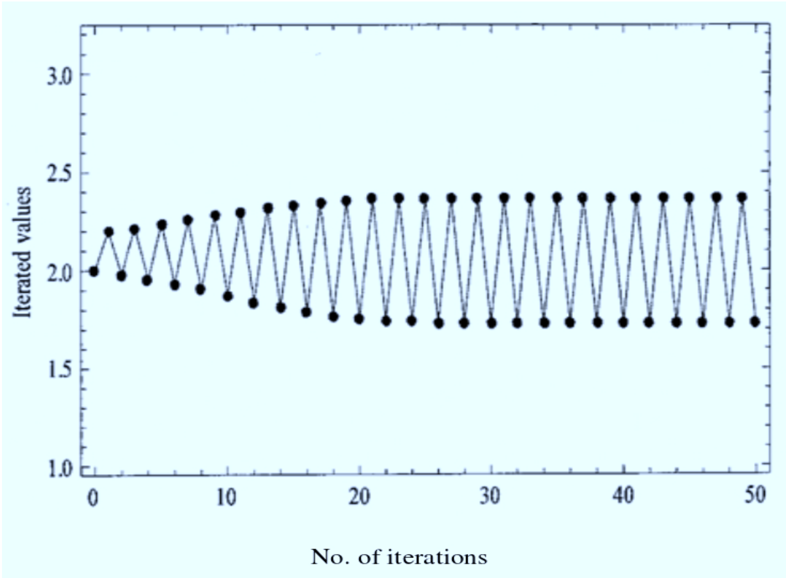


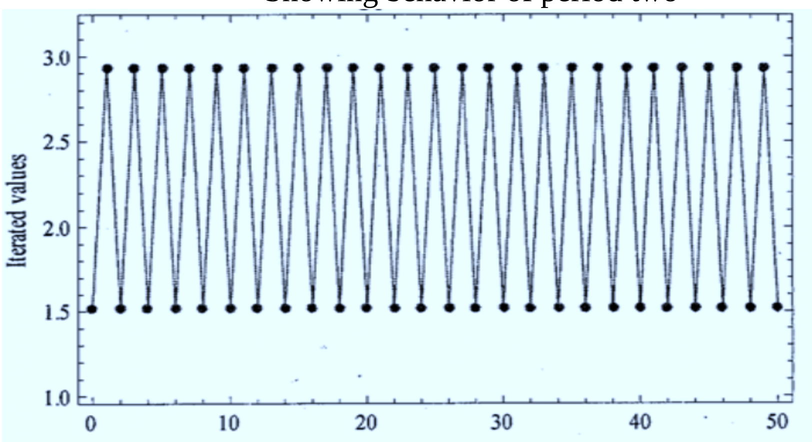
Figure 1.3: The time Series Showing Behavior of Period Two

Graphs of Time Series

Both periodic and bifurcation points are shown on the time series graph. The symbols B.P. and P.Ps. in the following figures represent the bifurcation point and periodic points, respectively.

Parameter value d=2nd B.P. Initial point one of P.Ps.

Showing behavior of period two

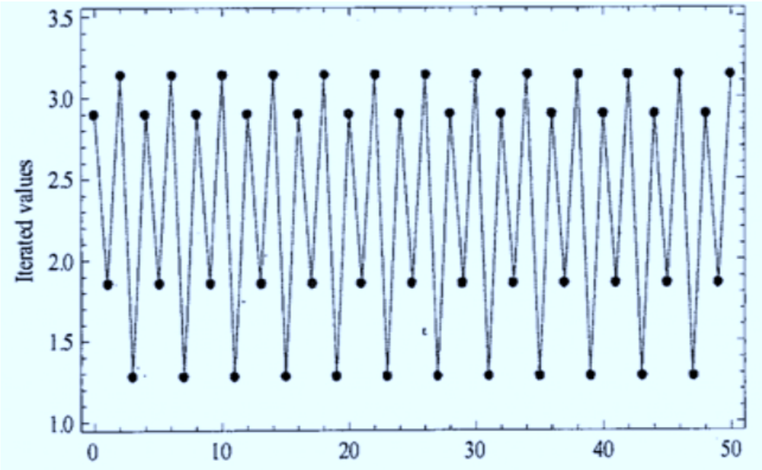


No. of iterations

Figure-1.4

Parameter value d=3rd B.P. Initial point one of P.Ps.

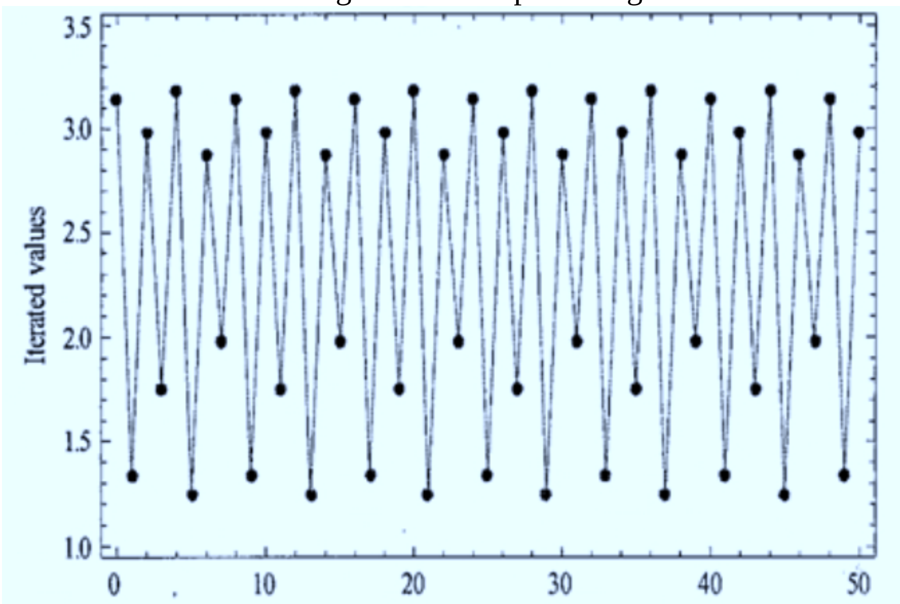
Showing behavior of period four



No. of iterations

Figure-1.5

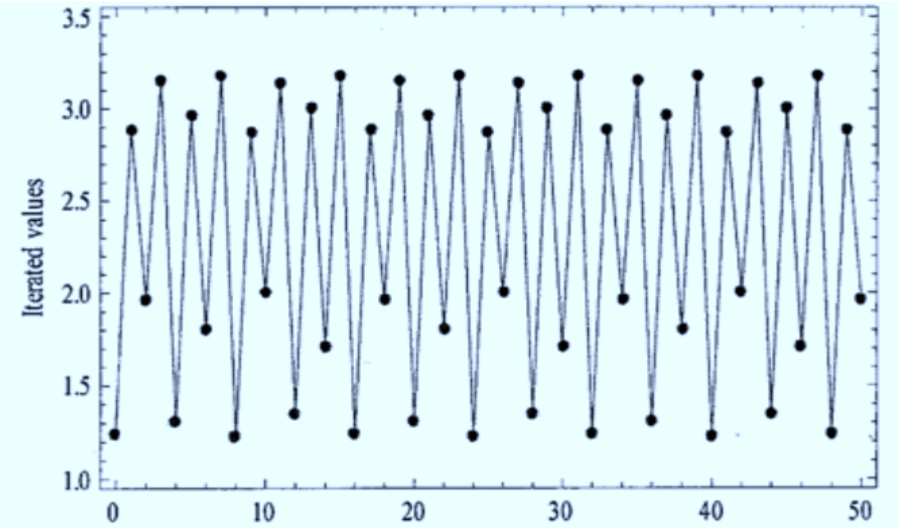
Parameter value d=4th B.P. Initial point one of P.Ps.
Showing behavior of period eight



No. of iterations

Figure-1.6

Parameter value d=5th B.P. Initial point one of P.Ps.
Showing behavior of period sixteen

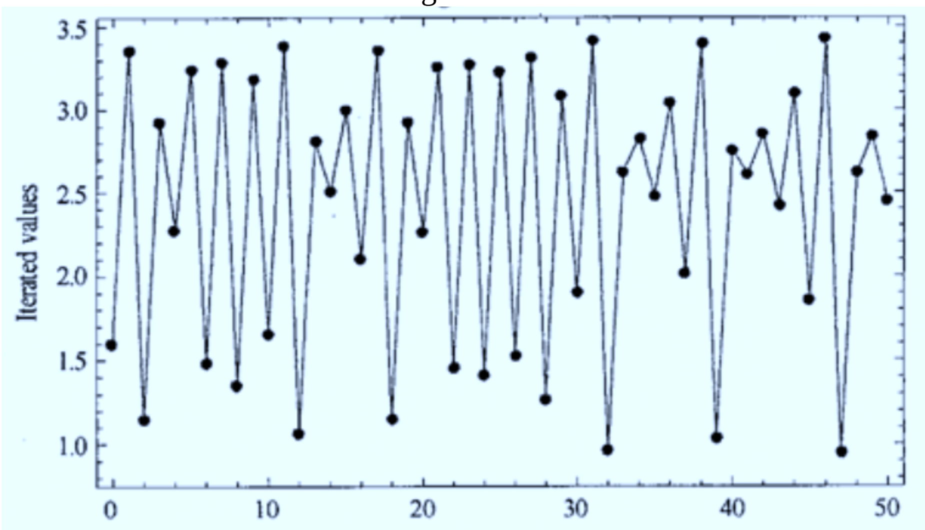


No. of iterations

Figure-1.7

Parameter value d=3.7, Initial point x=1.6

Showing chaotic behavior



No. of iterations

Figure-1.8

Conclusions

With an initial nonzero value of x, consecutive points converge to a fixed point if we begin with a value of d that is slightly smaller than d_1. However, for values of d that are marginally higher than d_1, the fixed point "bifurcates," creating a period-2 periodic orbit. This splits once more, meaning that at a high value of d, the period doubles to a period- 4 periodic orbit, and so forth. In this manner, as d rises, the period keeps doubling at ever closer values of d until chaotic behavior results. It is possible to continue this phenomenon until d= -3.56994 (approx.). Then a chaotic attractor shows up.

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