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SOLUTIONS OF EINSTEIN-MAXWELL FIELDS FOR HIGHER DIMENSIONAL SPACE-TIME

Dr. Ramesh Kumar
(Research Scholar)
Registration No-XM/R/285/2022
Department of Mathematics
Magadh University Bodh-Gaya (Bihar)-824234, India

ABSTRACT:

In the present paper we have deals with interior solutions for spherical symmetry in Einstein-Maxwell theory for higher dimensions.

Here we have found various solutions in different cases. The parameters appearing in these solutions can be evaluated by matching the solutions to the exterior Reissner-Nordstrom metric in higher dimensions. The higher dimensional spherically symmetric metric is taken in the form

$$ds^2 = e^B t(dt)^2 - e^A (dr)^2 - r^2[(d\theta_1)^2 + sin^2\theta_1 (d\theta_2)^2$$

$$+......+\sin^2\theta_1.....\sin^2\theta_n d\theta_{n+1}^2]$$

where A and B are functions of r only. Here

$$t = x^0, r = x^1, \theta_1 = x^2, \theta_2 = x^3......$$

Key Words: dimension, space-time, spherical symmetry, universe, field equation.

1. INTRODUCTION

Solutions of spherically symmetric charged (and uncharged also) fluid spheres have been studied by various authors in general Relativity using different conditions on metric functions or taking suitable form of charge Q or taking suitable relation between pressure and density and their physical and geometrical properties have been studied and discussed. However very less work has been done in case of charged fluid spheres in higher dimensions. Here we focus our attention to the study of charged fluid spheres in higher dimensions.

As a matter of fact, the recent developments in superstring theories show the requirement of the background space-time to be of 1 + 9 dimension (Schwarz [14] and Weinberg [15]) and also in view of Kaluza-Klein theory the higher dimensional physics has become important to study (Emelyanov *et. al* [6]). The existence of compact astrophysical objects in higher dimensions provide many arguments including the idea of a "gravitational

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bag" Which shows a higher dimensional configuration of the gravitational field (Guendelman and Owen [7]. Higher dimensional black holes have been investigated by Myers and Perry [9], Myers and Simon [10] and Xu Dianyan [20]. The early universe shows existence of domains of different dimensions (Sakharov [11]). Also, they communicate with the four four-dimensional space time we live in by electromagnetic effect [11]. The propagation of electromagnetic waves in higher dimensions was studied by Sokolowsci *et.al.* (12, 13]. They found the effective frequency in four dimensions from the higher dimensional frequency. These investigations show the possibility of higher dimensional stable compact object and can be detected in principle. In this way the study of charged fluid sphere in higher dimensions are significant and also interesting as a possible source of extra galactic radiation. Some other workers in this field are Abbott *et al*. [1]. Adams *et al*. [2]. Accetta and Gleiser [3] Brito *et. al.* [4], Bhar and Ratanpal [5], Krori *et al.* [8] Wiltshire [16], Whitman [18] and Xingxiang [19].

Here in this paper, we have investigated interior solutions for spherical symmetry in Einstein-Maxwell theory for higher dimensions. We have found various solutions in different cases. The parameters appearing in these solutions can be evaluated by matching the solution to the exterior Reissner-Nordstrom metric in higher dimensions.

2. THE FIELD EQUATIONS

The higher dimensional spherically symmetric metric is taken in the form

(2.1) $ds^2 = e^B t(dt)^2 - e^A(dr)^2 - r^2[(d\theta_1)^2 + sin^2\theta_1 (d\theta_2)^2$

$+ \ldots\ldots + \sin^2\theta_1 \ldots\ldots \sin^2\theta_n d\theta_{n+1}^2]$

where A and B are functions of r only.

Here

$t = x^0, r = x^1, \theta_1 = x^2, \theta_2 = x^3 \ldots\ldots$

and D = n +3 is the dimension of the space-time and 0 < r < ∞ , 0 < q1 < p, 0 < f < 2p. For D = 4 it provides usual spherically symmetric line element of four dimensions. The components of the Einstein tensor G_i^j given by

(2.2) $G_i^j = R_i^j - \frac{1}{2} R \delta_i^j$

Are found to be for the metric (2.1) as

(2.3) $G_0^0 = e^A \left[-\frac{(D-2)A'}{2r} + \frac{(D-2)(D-3)}{2r^2} - \frac{(D-2)(D-3)}{2r^2} \right]$

(2.4) $G_1^1 = e^A \left[\frac{(D-2)B'}{2r} + \frac{(D-2)(D-3)}{2r^2} - \frac{(D-2)(D-3)}{2r^2} \right]$

(2.5) $G_2^2 = G_3^3 = \ldots\ldots = G^{D-1}D - 1 = e^A \left[\frac{1}{2} B'' + \frac{1}{4}(B')^2 - \frac{1}{4} A'B' \right.$

$\left. - \frac{(D-3)(A'-B')}{2r} - \frac{(D-3)(D-4)}{2r^2} \right] - \frac{(D-3)(D-4)}{2r^2}$

The energy-momentum tensor for matter is taken as

(2.6) $T_0^0 = \rho, T_1^1 = p_r$

$T_2^2 = T_3^3 = \ldots\ldots = T^{D-1} D_{-1} = P_\perp$

where r is energy density, P_r is radial pressure and $P_\perp$ is tangential pressure.

The electromagnetic energy tensor E_{ij} is given by

(2.7) $E_{ij} = \frac{1}{\alpha} \left[g_{ij}(F_{ab}F^{ab}) - 4F_{ia}F_j^a \right]$

were

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$$\alpha = \frac{4\pi^{(D-1)/2}}{\left(\frac{D-1}{2}\right)!}(D-1)$$

For D = 4, a = 16p.

The Maxwell equation is

(2.8) $\frac{\delta}{\delta x^j}\left(\frac{4}{\alpha}F^{ij}\sqrt{-g}\right) = j^i\sqrt{-g}$

In the space time (2.1), the Maxwell equation (2.8) gives

(2.9) $\frac{d}{dr}\left[\sqrt{-g}F^{01}/\alpha\right] = 4\pi\sigma_0 u^0\sqrt{-g}$

where s0 is the charge density and

(2.10) $u^i = \delta_0^i (g_{00})^{-\frac{1}{2}}$

If we set F_{10} = E® and $\sigma_0 e^{-A/2} = \bar{\sigma}$

The equation (2.9) goes to the form

(2.11) $\frac{d}{dr}\left[\frac{4}{\alpha}e^{-(A+B)^{11/2}}E(r)r^{D-2}\right] = r^{D-2}\bar{\sigma}$

Integration of (2.11) provides

(2.12) $E(r) = \frac{e^{(A+B)/2}}{r^{D-2}}\int_0^r \frac{\alpha}{4}\bar{\sigma}r^{D-2}dr$

The total charge within D-dimensional sphere is taken as

(2.13) $Q(r) = \frac{\alpha}{4}\int_0^r \bar{\sigma}r^{D-2}dr$

Using this, equation (2.12) reduces to

(2.14) $E(r) = e^{-(A+B)/2} = \frac{Q(r)}{r^{D-2}}$

Using above equations, the Einstein Maxwell's equations can be cast to the forms

(2.15) $e^{-A}\left[\frac{(D-2)A'}{2r} - \frac{(D-2)(D-3)}{2r^2}\right] + \frac{(D-2)(D-3)}{2r^2} = \frac{2Q^2}{kr^{2D-4}} + \rho$

(2.16) $e^{-A}\left[\frac{(D-2)B'}{2r} + \frac{(D-2)(D-3)}{2r^2}\right] - \frac{(D-2)(d-3)}{2r^2} = \frac{-2Q^2}{kr^{2D-4}} + p_r$

(2.17) $e^{-A}\left[\frac{1}{2}B'' + \frac{1}{4}(B')^2 - \frac{1}{4}A'B' - \frac{(D-3)(A'-B')}{2t} + \frac{(D-3)(D-4)}{2r^2}\right] - \frac{(D-3)(D-4)}{2r^2} = \frac{2Q^2}{kr^{2D-4}} + p_\perp$

Where a prime denotes differentiation w.r.t r.

3. SOLUTIONS OF THE FIELD EQUATION

Here we consider case of perfect fluid so that p_r = $p_\perp$ and we have three equations (2.15) – (2.17) to find five unknows A, B, p, r and Q. So, the system is indeterminate. For complete determinacy of the system, we require two more conditions or relations.

Equation (2.15) can be written as

(3.1) $\frac{d}{dr}[r^{D-3}e^{-A}] = (D-3)r^{D-4} - \frac{2\rho}{(D-2)}r^{D-2} - \frac{4Q^2}{\alpha(D-2)r^{D-2}}$

By use of equations (2.16) and (2.17) we find

(3.2) $$e^{-A}\left[\frac{B''}{2} + \frac{1}{4}(B')^2 - \frac{1}{4}A'B' - \frac{[B'+(D-3)A']}{2r} - \frac{(D-3)}{r^2}\right] + \frac{D-3}{r^2} - \frac{4Q^2}{\alpha r^{2D-4}} = 0$$

To solve the equations (2.5) (3.2), we consider the following different cases for perfect fluids (i.e. $p_r = p_\perp = p$) which are physically interesting.

Case – I

Here we assume

(3.3) $\rho = \rho_0 r^N$

For general value of N and

(3.4) $Q(r) = Q_0\left(\frac{r}{r_0}\right)^{D-1}$

Where Q_0 is the total charge within a sphere of radius r_0. Now to avoid mathematical complexity, we consider the following solutions for special cases of N.

Case II

(3.17) $\rho = \rho_0\left(1 - \frac{r^2}{r_0^2}\right)^L$

where L is +ve integer and r_0 is radius of sphere. To avoid complex mathematical calculation, we choose L = 1, so that condition (3.17) reduces to the form

(3.18) $\rho = \rho_0\left(1 - \frac{r^2}{r_0^2}\right)$

Also,

$$Q = \frac{Q_0 r^{D-1}}{r_0^{D-1}}$$

In this case equation (3.1) immediately provides in solution

(3.19) $e^{-A} = 1 - \frac{2\rho_0 r^2}{(D-1)(D-2)} + \frac{2r^4}{(D+1)(D-2)}\left[\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right]$

Further taking $r^2 = x$ and $B = 2\log\Psi$ and using equation (3.7), from equation (3.14) we obtain

(3.20) $$\left[1 - \frac{2\rho_0 x}{(D-1)(D-2)} + \frac{2x^2}{(D+1)(D-2)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right)\right]\frac{d^2\psi}{dx^2} - \left[\frac{\rho_0}{(D-1)(D-2)} - \frac{2x}{(D+1)(D-2)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right)\right]\frac{d\psi}{dx} + \left[\frac{1}{2(D+1)(D-2)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right) - \frac{Q_0^2}{\alpha r_0^{2D-2}}\right]y = 0$$

which by using the substitution

(3.21) $$\xi = \frac{\left[\frac{1}{2(D+1)(D-2)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right) - \frac{Q_0^2}{\alpha r_0^{2D-2}}\right]^{1/2}}{\left[\frac{2}{(D+1)(D-2)}\left(\frac{\rho}{r^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right)\right]^{1/2}} cosh^{-1}\left[\frac{\left\{\frac{2x}{(D+1)(D-2)}\left(\frac{\rho_0}{2} - \frac{2Q_0^2}{r_0^{2D-2}}\right) - \frac{\rho_0}{(D-1)(D-2)}\right\}}{\left\{\frac{\rho_0^2}{(d-1)^2(D-2)^2} - \frac{2}{(D+1)(D-2)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right)\right\}^{1/2}}\right]$$

Goes to the form

(3.22) $\frac{d^2\psi}{d\varsigma^2} + \psi + 0$

The solution of (3.22) may be written as

(3.23) $\psi = M\,sin\,\varsigma + \eta\,cos\,\varsigma$

or

$$e^B = [M\,sin\,\varsigma + \eta\,cos\,\varsigma]^2$$

Using equations (3.19) and (3.23) in equation (2.16) we obtain

$$(3.24)\ p = \frac{2Q_0^2 r^2}{\alpha r_0^{2D-2}} + (3-D)\left[\frac{\rho_0}{(D-1)} - \frac{r^2}{(D+1)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right)\right] + \frac{2(D-2)(D\,cos\,\varsigma - \eta\,sin\,\varsigma)}{(M\,sin\,\varsigma + \eta\,cos\,\varsigma)}$$

$$\left[1 - \frac{2\rho_0 r^2}{(D-1)(D-2)} + \frac{2r^4}{(D+1)(D-2)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{2D-2}\right)\right]^{1/2}$$

$$\left[\frac{1}{2(D+1)(D-2)}\left(\frac{\rho_0}{r_0^2} - \frac{2Q_0^2}{r_0^{2D-2}}\right) - \frac{Q_0^2}{\alpha r_0^{2D-2}}\right]^{1/2}$$

All the above solutions can be easily matched at the boundary r = r_0 with the D-dimensional Reissner-Nordstrom solution for the total charge Q_0 within a fluid sphere of radius r_0. This is the solution given by Wolf [17]

$$(3.25)\ e^B = e^{-A} = \frac{1 - 2\varsigma\bar{m}k_2(D)}{r^{D-3}k_1^2} + \frac{32\pi\varsigma Q_0^2}{k_1^4 \alpha (D-2)(D-3) r^{2D-6}}$$

Were

$$(3.26)\ \begin{cases} k_2(D) = \dfrac{8\pi}{(D-2)C_{D-2}} \\ C_{D-2} = (D-1)\dfrac{\pi^{(D-1)/2}}{\left(\frac{D-1}{2}\right)!} \end{cases}$$

Case: III

Here we choose a relation between p and r given as follows

(3.27) p + r = 0

and Q as case I & II

Using the relation (3.27) in equations (2.15) and (2.16), we get

(3.28) A = – B

Using equations (3.27), (3.28) and putting $e^{-\lambda} = \tau, r = e^{\psi}$, equation (3.2) can be transformed to

$$(3.29)\ \frac{d^2\tau}{d\psi^2} + (D-5)\frac{d\tau}{d\psi} - 2(D-3)\tau = \frac{8Q_0^2 e^{-4\psi}}{\alpha r_0^{2(D-1)}} - 2(D-3)$$

Solution of differential equation (3.29) may be written as

$$(3.30)\ \tau = 1 + k_1 r^{2\psi} + k_2 e^{(3-D)\psi} + \frac{4Q_0^2 e^{-4\psi}}{3(7-D)\alpha r_0^{2(D-1)}}$$

or

$$e^{-A} = 1 + k_1 r^2 + k_2 r^{(3-D)} + \frac{4Q_0^2 r^{-4}}{3\alpha(7-d) r_0^2 D - 1}$$

By the using of equation (2.15), (3.27) and (3.30), we find the density r as

$$(3.31)\ \rho = \left[\frac{(2-D)}{2}(D-1)K_1 + (4-D)k_2 r^{1-D} + \frac{20D_0^2 r^{-6}}{3(7-D) r_0^{2(D-2)}}\right] - \frac{4Q_0^2 r^2}{\alpha r_0^{2(D-1)}}$$

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4. DISCUSSION

Here in this chapter, we have obtained the explicit solutions for charged perfect spheres in higher dimensions in different special cases. These solutions can be taken as the higher dimensional analogues of the known solutions of Einstein-Maxwell equations for four-dimensional charged fluid spheres.

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