



Function Spaces and Absolute Almost-Convergence

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Abstract

The objective of this paper is to introduce the new concept of absolute almost convergence in functional case. We generalize the definition given by Das and Nanda in [2], and also the certain new function spaces of Nanda in [10].

1. Introduction

Throughout this paper, E denote a Banach space of measurable essentially bounded real-valued functions defined on the real line R with the norm

$$\|f\| = \text{ess.sup. } \{|f(x)| : x \in R\}$$

and let L be the set of all the Lebesgue integrable functions in every finite interval , for each $x \geq 0$ and $f \in E$, we define the operator $T_{k,x}$ for each $t \in R$ by

$$(f + x); \quad k > 0, x \geq 0 \quad \dots(1.1)$$

$$T_{k,x}f(t) = \begin{cases} \frac{(f + x)}{k} \int_t^{t+x} (t + x - u)^{k-1} f(u) du, & k > 0, x > 0, \\ \frac{1}{x^k} & \end{cases}$$

$$T_{k,x}f(t) = \frac{k}{x^k} \int_0^x (x - v)^{k-1} f(v + t) dv \quad \dots(1.2)$$

We assume that E contains the set S of bounded uniformly continuous function on R and the functions $T_{k,x}f$ for each $f \in E$ and $x \geq 0$. Let

$$F_k = \{f \in E : \lim_{x \rightarrow \infty} T_{k,x} f(t) \text{ exists uniformly in } t\} \quad \dots (1.3)$$

2. Key Words and Phrases

Almost convergence, strongly F_k - almost convergence, Absolute F_k - almost convergence, function spaces.

We call F_k , the set of all almost convergent functions and in this case we write

$$Fk - \lim f(t) = \lim_{x \rightarrow \infty} T_{k,x} f(t) \text{ (see [5] \& [11])} \quad \dots(1.4)$$

Note that if $k = 1$ the $F_1 = F$ (see [2]) where F is the continuous analogue of almost convergent sequences (see [6] and [1]).

We introduce $[F_k]_p$, the set of all strongly F_k - almost convergent functions with index p :

$$[F_k]_p = \{f : \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x |T_{k-1,u} f(t) - s|^p du = 0\} \quad \dots(1.5)$$

for some s uniformly in t } $k \geq 1, p \geq 1$

It is evident that $[F_k]_p \subset F_k$ and $[F_k]_p - \lim f = s \Rightarrow F_k - \lim f = s$. (see [7] \& [8]).

Let $f(u)$ be any function which is Lebesgue integrable in $(0, u)$ for all finite $u > 0$, and that $f(u)$ is bounded in some right hand neighborhood of origin.

$$\text{Let } f(u) = \int_0^u a(w)dw \quad (1.6)$$

where $a(\omega)$ is Lebesgue integrable in each finite interval of $u \geq 0$. Then it is easy to justify the differentiation under the integral sign and that

$$x \frac{d}{dx} T_{k,x} f(t) = k [T_{k-1,x} f(t) - T_{k,x} f(t)] = \frac{k}{x^k} \int_0^x (x-v)^{k-1} v a(v+t) dv = \phi_k(x,t), \text{ say } \dots(1.7)$$

Dividing out by x , we get

$$\frac{d}{dx} T_{k,x} f(t) = \frac{k}{x^{k+1}} \int_0^x (x-v)^{k-1} v a(v+t) dv = \phi_k(x,b) \quad \text{say (1.8)}$$

3. Absolute Almost Convergence

(i) We define $|F_k|_p$, for $p \geq 0$ and $k \geq 1$, the set of all absolutely F_k -almost convergent functions with index p :

$$|F|_p = \left\{ f : \int_T^\infty x^{p-1} \left| \frac{d}{dx} T_{k,x} f(t) \right|^p dx < \infty \right\} \text{ uniformly in } t \text{ for some } T \geq 0^+ \} \quad \dots(2.1)$$

In view of relation (1.7), we may restate the definition $|F_k|_p$ as follows :

$$|F_k|_p = \left\{ f : \int_T^\infty x^{p-1} |\phi_k^x(x,t)|^p dx < \infty \text{ uniformly in } t \text{ for some } T \geq 0 \right\} \quad \dots (2.2)$$

In view of relation (1.8), we may restate the definition $|F_k|_p$ as follows :

$$|F_k|_p = \left\{ f : \int_T^\infty x^{p-1} |\phi_k(x,t)|^p dx < \infty \text{ uniformly in } t \text{ for some } T \geq 0 \right\} \quad \dots (2.3)$$

We further define

$$|\hat{F}_k|_p = \left\{ f : \sup_t \int_T^\infty x^{p-1} |\phi_k(x,t)|^p dx < \infty \right\} \quad \dots (2.4)$$

Note that, in any results involving the space $|F_k|_p$ for values of $k < 1$ we restrict ourselves to the case in which $f(u)$ is an indefinite integral of a function $a(u)$, say: \sup ensures that the derivative $\left(\frac{d}{dx} T_{k,x} f(t) \right)$ exist almost everywhere.

Such a restriction is not, however, needed when $k \geq 1$.

In the case $p = \infty$, we interpret

$$\int_T^\infty x^{p-1} \left| \frac{d}{dx} T_{k,x} f(t) \right|^p dx < \infty,$$

as meaning

$$x \frac{d}{dx} T_{k,x} f(t) = 0(1)$$

$$\text{Hence } |F_k|_\infty = \left\{ f : x \frac{d}{dx} T_{k,x} f(t) = 0(1) \text{ uniformly in } t \right\} \quad \dots(2.5)$$

It is evident that $|F_k| \subset F_k$ but the converse inclusion is not always true. However the inclusion $|F_k|_p \subset F_k$, does not, in general hold.

We also define the spaces

$$[F_k]_p = \left\{ f : \lim_{x \rightarrow \infty} x^{-1} \int_0^{x^1} \left| x \frac{d}{dx} T_{k,x} f(t) \right|^p dx < \infty \text{ uniformly in } t \right\} \quad \dots(2.6)$$

$$[\vec{F}_k]_p = \left\{ f : \int_1^\infty x^{-1} |T_{k,x} f_{f(t)-s}^{(t)-s}|^p dx < \infty \text{ uniformly in } t \right\} \quad \dots (2.7)$$

It is worth mentioning that (as may easily be proved by integration by parts)

$$[\vec{F}_k]_p \subset [F_k]_p$$

(ii) For $p \geq 1, k = 1$

$$[F_1]_p = [F]_p = \{f : \int_1^\infty x^{p-1} \left| \frac{d}{dx} T_{1,x} f^{f(t)} \right|^p dx < \infty \text{ uniformly in } t \text{ for some } T \geq 0\} \dots(2.8)$$

$$= \{f : \int_T^\infty x^{p-1} |\phi(x,t)|^p dx < \infty \text{ uniformly in } t \text{ for some } T \geq 0\}$$

$$[\hat{F}_1]_t = \{f : \sup_t \int_T^\infty x^{p-1} |\phi(x,t)|^p dx < \infty\} \dots(2.9)$$

where $T_{1,x}, f(t) = \frac{1}{x} \int_0^x f(v+t)dv$

In this case

$$\begin{aligned} x \frac{d}{dx} T_{1,x} f(t) &= \{T_{0,x} f(t) - T_{1,x} f(t)\} \\ &= \left[f(x+t) - \frac{1}{x} \int_0^T f(v+t)dv \right] \end{aligned}$$

Dividing out by x , we have

$$\frac{d}{dx} T_{1,x} f(t) = \frac{1}{x^2} \int_0^x [f(x+t) - f(v+t)]dv = \phi(x,t) \text{ say [see[2]]}$$

We now give a generalization of the set of absolute almost convergent functions defined by Das and Nanda (in [2]. we define

$$\hat{L}_k = \{f : T_{k,x} f(t) \in BV_x \text{ uniformly in } t\} \dots (2.10)$$

Since $T_{k,x} f(t) \in BV_x$ uniformly in t implies

$T_{k,x} f(t) \rightarrow$ a limit as $x \rightarrow \infty$ uniformly in t , it follows that $\hat{L}_k \subset F_k$. In view of relation (1.8), we may restate the definition of \hat{L}_k as follows.

$$\hat{L}_k = \{f : \int_1^\infty |\phi_k(x,t)| dx \text{ converges uniformly in } t\} \dots (2.11)$$

$$\hat{L}_k = \{f : \sup_t \int_1^\infty |\phi_k(x,t)| dx < \infty\} \dots (2.12)$$

In further $f \in BV$. then We have,

$$\phi_k(x,t) = \frac{k}{x^{k+1}} \int_0^x (x-v)^{k-1} v df(v+t) = \frac{k}{x^{k+1}} \int_0^x (x-v)^{k-1} va(v+t)dv \dots(2.13)$$

where $\frac{d}{du} f(u) = a(u)$ a.e.

(iii) For $0 < p < \infty, k > 1$

$$\hat{L}_{k,p} = \{f : \int_1^\infty |\phi_k(x,t)|^p dx \text{ converges uniformly in } t\} \dots (2.14)$$

$$\hat{L}_{k,p} = \{f : \sup_t \int_1^\infty |\phi_k(x,t)|^p dx < \infty\} \dots (2.15)$$

Note that if $p=1$ then $\hat{L}_{k,p} = \hat{L}_k$ & $\hat{L}_{k,p} = \hat{L}_k$

The cases $k=1$ of definitions (2.10) to (2.15) are due to Das and Nanda [2]. The series analogues of these concepts have been introduced and discussed in [3]

4. Some Function Spaces

We define, for $1 \leq p < \infty$

$$\bar{L}_{k,p} = \{f : \int_1^\infty |T_{k,x}f(t)|^p dx \text{ converges uniformly in } t\}$$

We introduce another related concept

$$\bar{\bar{L}}_{k,p} = \{f : \sup_t \int_1^\infty |T_{k,x}f(t)|^p dx < \infty\}$$

$$|C_k|_p = \{f : \int_1^\infty \left| \frac{k}{x^k} \int_0^x (x-v)^{k-1} f(v)dv \right|^p dx < \infty\}.$$

$$\bar{L}_{k,\infty} = \{f : \sup_{t,x} |T_{k,x}f(t)| < \infty\}.$$

Note that

$$L_p = \{f : \int_0^\infty |f(t)|^p dt < \infty\}, \text{ and } L_\infty = \{f : \text{ess sup } |f(t)| < \infty\}$$

Further we see that it is evident that $\bar{L}_{kp} \subset |C_k|_p$

$$|T_{k,x}f(t)| \geq |T_{0,x}f(t)| = |f(t+x)|, x \geq 0, k = 0 \text{ and}$$

and
$$|T_{k,x}f(t)| \leq \frac{k}{x^k} \sup_v |f(v+t)| \int_0^x (x-v)^{k-1} dv$$

$$= \sup_v |f(v+t)| = \sup_t |f(t)|$$

and hence $\bar{L}_{k,\infty} = L_\infty$

Further we define

$$|C_{k,s}|_p = \{f : \int_1^\infty [\frac{k}{x^k} \int_0^x |f(v)|^p dv] dx < \infty\}$$

then evidently $|C_{k,s}|_p \subset |C_k|_p$

Theorem (4.1): $\bar{L}_{k,p} \subset \bar{\bar{L}}_{k,p}$

Proof.: Let $f \in \bar{\bar{L}}_{k,p}$. This implies that

$$\int_1^\infty |T_{k,x}f(t)|^p dx < \infty, \text{ uniformly in } t.$$

$$\Rightarrow \sup_t \int_1^\infty |T_{k,x}f(t)|^p < \infty$$

Hence $f \in \bar{\bar{L}}_{k,p}$.

Theorem (4.2): For $1 \leq p < \infty, k \geq 1, L_p \subset \bar{L}_{k,p}$

In order to prove this theorem, we require the following inequality.

Lemma (Inequality). For $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \int_0^\infty |f(x)|^p dx < \infty$ we have

$$\int_0^\infty [\frac{1}{x} \int_0^x |f(t)| dt]^p dx < q^p \int_0^\infty |f(t)|^p dt.$$

See Hardy, Littlewood and Polya [(92) Chapter 9] for the proof of above inequality.

Proof of Theorem 4.2 : Let $f \in L_p$. It follows trivially for $p = 1$ and from the above inequality

for $p > 1$ that
$$\int_0^\infty [\frac{k}{x^k} \int_t^{t+x} (t+x-u)^{k-1} |f(u)| du]^p dx = \int_0^\infty [\frac{k}{x^k} \int (x-v)^{k-1} |F(v+t)| du]^p$$

$$< q^p \int_1^\infty |f(v+t)|^p dv = q^p \int_1^\infty |F(v)|^p dv < \infty.$$

Hence $f \in \bar{L}_{k,p}$, and this completes the proof.

Theorem 4.3 : For $1 \leq p < \infty, k \geq 1$, \bar{L}_p and $\bar{\bar{L}}_{k,p}$ are Banach spaces normed by

$$\|f\|_{\hat{p},k} = \sup_t \left(\int_1^\infty |T_{k,x} f(t)|^p dx \right)^{\frac{1}{p}} \quad \dots(4.1)$$

Proof : Let $f \in \bar{L}_{k,p}$. Then (4.1) is meaningful because of theorem 4.3. It is fairly easy to verify that (4.1) defines a norm. In the same way to the case of L_p , it can be shown that $\bar{L}_{k,p}$ and $\bar{\bar{L}}_{k,p}$ are complete in the norm.

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