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Almost α-duals of generalized difference sequence spacs $l_\infty(\Delta^m)$ and $l_\infty(\Delta_v)$

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Abstract

In this paper using the concept of absolutely almost convergence introduced by Das, Kuttner and Nanda [4] we have determine the concept of almost α-dual in [2] and have determined the same for generalized difference sequence space $l_\infty(\Delta^m)$ and $l_\infty(\Delta_v)$.

1 Introduction

Let l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)_{k=1}^{\infty}$ of complex numbers respectively, normed by

$$\|x\|_\infty = \sup_{k\geq 1} |x_k|$$

where $k \in N$ and N is the set of positive integers.
Kizmaz [6] defined the sequence spaces

$$l_\infty(\Delta) = \{x = (x_k): \Delta x \in l_\infty\}$$

$$c(\Delta) = \{x = (x_k): \Delta x \in c\}$$

and

$$c_0(\Delta) = \{x = (x_k): \Delta x \in c_0\}$$

where

$$\Delta x = (\Delta x_k) = (x_k - x_{k+1})$$

These spaces are Banach spaces with the norm

$$\|x\| = |x_1| + \|\Delta x\|_\infty$$

where

$$\|\Delta x\|_\infty = \sup_{k\geq 2} |x_k - x_{k+1}|$$

For convenience, we denote these spaces $\Delta l_\infty, \Delta c$ and Δc_0 and call the constituent sequences Δ-bounded, Δ-convergent and Δ-null sequences respectively.

Let E be any of the spaces ℓ_∞, c and c_0 then it is easy to see that $E \subset \Delta E$. Further, the containment is strict.

For example, let $x_k = k, k = 1,2,3, \dots$ then the sequence $(x_k) \notin c$ though $(x_k) \in \Delta c$.

Let $p = (p_k)$ denote a sequence of strictly positive numbers (not necessarily bounded). Ahmad and Mursaleen [1] defined the sequence spaces

$$\Delta\ell_\infty(p) = \{x = (x_k): \Delta x \in \ell_\infty(p)\}$$
$$\Delta c(p) = \{x = (x_k): \Delta x \in c(p)\}$$
$$\Delta c_0(p) = \{x = (x_k): \Delta x \in c_0(p)\}$$

When (p_k) is constant with all terms equal to $p > 0$, we have

$$\Delta\ell_\infty(p) = \ell_\infty(\Delta), \Delta c(p) = c(\Delta) \text{ and } \Delta c_0(p) = c_0(\Delta).$$

Et. and Colak [5] defined the sequence spaces

$$\ell_\infty(\Delta^m) = \{x = (x_k): \Delta^m x \in \ell_\infty\}$$
$$c(\Delta^m) = \{x = (x_k): \Delta^m x \in c\}$$

and

$$c_0(\Delta^m) = \{x = (x_k): \Delta^m x \in c_0\}.$$

where $m \in N$,

$$\Delta^0 x = x_k, \Delta x = (x_k - x_{k+1})$$
$$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$$

These spaces are Banach spaces with the norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty$$

where

$$\|x\|_\infty = \sup_{k>m} |\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}|$$

The idea of the dual sequence spaces was introduced by Köthe-Toeplitz [7] whose main results concerned α-duals. Since there are several type of duals in the literature. The α-dual of $E \subset S$ where S is the linear space of complex sequence and E denote a set or space of complex sequences is defined as

$$E^\alpha = \left\{a = (a_n) \in S: \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{for all } x \in E\right\}$$

Lorentz [8] has defined a sequence $x \in \ell_\infty$ to be almost convergent if all the Banach limit of x coincided. Let $\hat{C}$ denote the set of all almost convergent sequences. Lorentz [8] proved that

$$\hat{C} = \left\{x: \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} x_{n+i} \text{ exists uniformly in } n\right\}$$

In 1984 Das, Kuttner and Nanda [4] developed the concept of absolutely almost convergent series as an absolute analogue of almost convergence.

Ansari & Shukla [2] introduced the concept of almost α-dual and determined the almost α-dual of c_0, c and l_∞ namely,

$$c_0^{\hat{\alpha}} = c^{\hat{\alpha}} = \ell_\infty^{\hat{\alpha}} = \hat{\ell}_1$$

where

$$\hat{\ell}_1 = \left\{a = (a_k): \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i|a_{n+i}| < \infty\right\}$$

uniformly in n, which is the space of most absolutely almost convergent series. It is an extension of the well known fact that

$$c_0^{\alpha} = c^{\alpha} = \ell_\infty^{\alpha} = \ell_1$$

Hence distinction has been made between an absolutely almost convergent series and a most absolutely almost convergent series.

2 Some Definitions

Let

$$a = \sum_{i=0}^{\infty} a_i$$

be an infinite series of complex numbers and $(x_n)_{n=0}^{\infty}$ be its sequence of partial sums i.e.

$$x_n = a_0 + a_1 + a_2 + \cdots + a_n.$$

Das, Kuttner & Nanda [4] define

$$d_{k,n} = d_{k,n}(x) = \frac{1}{k+1} \sum_{i=0}^{k} x_{n+i}, (k > 0, n \geq 0)$$

Considering

$$d_{0,n} = d_{0,n}(x) = x_{n-1}$$

Write for $k, n \geq 0$

$$\phi_{k,n} = \phi_{k,n}(a) = d_{k+1,n} - d_{k,n}$$

Then

$$\phi_{0,n} = a_n$$

i.e.

$$\phi_{k,n} = \frac{1}{k+1} \sum_{i=1}^{k} i a_{n+i} \; (k \geq 1)$$

Then the series a or the sequence x is said to be absolutely almost convergent if

$$\sum_{k=1}^{\infty} |\phi_{k,n]}|$$

converges uniformly in n as defined o in [4].
Using the notion of absolutely almost convergence developed by Das and Kuttner and Nanda in [4], we introduce the concept of almost α-dual in [2].

Thus, if E is a set or a space of sequence of complex numbers, the almost α-dual of E is denoted by $E^{\hat{\alpha}}$, defined by
$E^{\hat{\alpha}} = \left\{a = (a_k): \sum_{k=1}^{\infty} \frac{1}{k+1} \left|\sum_{i=1}^{k} ia_{n+i}x_{n+i}\right| < \infty, \forall x = x_k \in E \text{ uniformly in } n\right\}$
We make use of the following lemma whose proof can be found in Kizmaz [6].

Lemma 1.

$$\sup_k |x_k - x_{k+1}| < \infty, \text{ iff}$$

(i)

$$\sup_{k\geq 1} k^{-1}|x_k| < \infty \;\&$$

(ii)

$$\sup_k |x_k - k(k+1)^{-1}x_{k+1}| < \infty$$

Theorem 1. The almost α-dual of $\ell_\infty(\Delta)$ is given by
$[\ell_\infty(\Delta)]^{\hat{\alpha}} = \left\{a = (a_k): \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i(n+i)|a_{n+i}| < \infty, \text{uniformly for every } n\right\} = \hat{D}_1.$
Proof: Let $(x_k) \in \ell_\infty(\Delta)$ and $a \in \hat{D}_1$, then

$$\sum_{k=1}^{\infty} |\phi_{k,n}| = \sum_{k=1}^{\infty} \left|\frac{1}{k(k+1)} \sum_{i=1}^{k} ia_{n+i}x_{n+i}\right|$$
$$\leq \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i(n+i)|a_{n+i}| \frac{|x_{n+i}|}{(n+i)}$$
$$< \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sup_{1\leq i\leq k} \frac{|x_{n+i}|}{(n+i)} \sum_{i=1}^{k} i(n+i)|a_{n+i}|$$
$$< \sup_{i\geq 1} \frac{|x_i|}{i} \sum_{i=1}^{k} \frac{1}{k(k+1)} \sum_{i=1}^{k} i(n+i)|a_{n+i}|$$
$$< \infty \text{ for every } n. \text{ [By Lemma 1]}$$

Thus

$$\sum_{k=1}^{\infty} |\phi_{k,n}| < \infty \;\text{ for all } n$$

Converse, suppose $a = (a_k) \in [\ell_\infty(\Delta)]^{\hat{\alpha}}$ but $(a_k) \notin \hat{D}_1$,
i.e.

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sum_{i=1}^{k} i(n+i)|a_{n+i}| = \infty \text{ for some } n.$$

Choose $x = (x_k)$ with $x_k = k\text{sgn}a_k$, then $x \in \ell_\infty(\Delta)$ but

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sum_{i=1}^{k} ia_{n+i} = \sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sum_{i=1}^{k} i(n+i)|a_{n+i}| = \infty$$

which is a contradiction to the fact that $(a_k) \in [\ell_\infty(\Delta)]^{\hat{\alpha}}$.
Hence

$$[\ell_\infty(\Delta)]^{\hat{\alpha}} = \hat{D}_1$$

Theorem 2: For every strictly positive sequence $p = (p_k)$, we have

$$[\Delta\ell_\infty(p)]^{\hat{\alpha}} = \hat{D}_1(p), \text{ where}$$

$$\hat{D}_1(p) = \bigcap_{N>1}\left\{a = (a_k): \sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sum_{i=1}^{k} i(n+i)|a_{n+i}|N^{1/p_{n+i}} < \infty \text{ uniformly in } n\right\}$$

Proof: Let $a \in \hat{D}_1(p)$ and $x \in \Delta\ell_\infty(p)$. We choose an integer

$$N > \max\left\{1, \sup_k \left|\frac{x_k}{k}\right|^{p_k}\right\}$$

Then

$$\sum_{k=1}^{\infty}\left|\frac{1}{k(k+1)}\sum_{i=1}^{k} ia_{n+i}x_{n+i}\right| \leq \sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sum_{i=1}^{k} i(n+i)|a_{n+i}|\frac{|x_{n+i}|}{(n+i)}$$
$$< \sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sum_{i=1}^{k} i(n+i)|a_{n+i}|N^{1/p_{n+i}} < \infty$$

therefore

$$(a_k) \in [\Delta\ell_\infty(p)]^{\hat{\alpha}}$$

Converse: Now suppose that

$$a \in [\Delta\ell_\infty(p)]^{\hat{\alpha}}$$

but a $\notin \hat{D}_1$, i.e., there is an integer $N > 1$ such that

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sum_{i=1}^{k} i(n+i)|a_{n+i}|N^{1/p_{n+i}} = \infty$$

Choose $x = (x_k)$ with $x_k = kN^{1/p_k}\text{sgn}a_k$. Then

$$x \in \Delta\ell_\infty(p) \text{ but}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i a_{n+i} x_{n+i} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i(n+i)|a_{n+i}| N^{1/p_n} = \infty$$

which is a contradiction that

$$(a_k) \in [\Delta\ell_\infty(p)]^{\hat{\alpha}}$$

Hence

$$[\Delta\ell_\infty(p)]^{\hat{\alpha}} = \hat{D}_1(p)$$

Theorem 3. Let $p_k > 0$ for every k, then

$$[\ell_\infty(p)]^{\hat{\alpha}} = \hat{\ell}_1(p)$$

where

$$\hat{\ell}_1(p) = \bigcap_{N=2}^{\infty} \left\{ a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i|a_{n+i}| N^{1/p_{n+i}} < \infty \text{ uniformly in } n \right\}.$$

Proof. Let $a \in \hat{\ell}_1(p)$ and $x \in \ell_\infty(p)$. We choose an integer

$$N > \max\left\{1, \sup_k |x_k|^{p_k}\right\}$$

then

$$\sum_{k=1}^{\infty} |\phi_{k,n}| = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i|a_{n+i}||x_{n+i}| \le \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i|a_{n+i}| N^{1/p_{n+i}} < \infty$$

and therefore

$$\hat{\ell}_1(p) \subset [\ell_\infty(p)]^{\hat{\alpha}}$$

Conversely, if $(a_k) \in [\ell_\infty(p)]^{\hat{\alpha}}$ but $(a_k) \notin \hat{\ell}_1(p)$, then there exists an integer $N > 1$ such that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i|a_{n+i}| N^{1/p_{n+i}} = \infty$$

Putting

$$x_k = N^{1/p_k} \mathrm{sgn} a_k$$

we have $x \in \ell_\infty(p)$, but

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i a_{n+i} x_{n+i} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i|a_{n+i}| N^{1/p_{n+i}} = \infty$$

which is a contradiction that $(a_k) \in [\ell_\infty(p)]^{\hat{\alpha}}$. Hence

$$[\ell_\infty(p)]^{\hat{\alpha}} = \hat{\ell}_1(p)$$

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