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Almost α-duals of c_0, c and l_∞ spaces & Semi Difference Sequence Spaces & their Köthe-Toeplitz Duals

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Abstract

In this paper we introduce the concept of almost α-dual and determine the same for the well known sequence spaces c_0, c and l_∞ which denote respectively the sequence space of null, convergent and bounded sequences of complex numbers.

1 Introduction

The idea of the dual sequence spaces was introduced by Köthe and Toeplitz [3] whose main results concerned α-duals. The α-dual of $E \subset S$ where S is the linear space of complex sequences and E denote a set or space of complex sequences is defined as

$$E^\alpha = \left\{a \in S: \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{for all } x \in E\right\}$$

Let c_0, c and l_∞ respectively denote the Banach space of all null, convergent and bounded sequences of real or complex numbers normed as usual by

$$\|x\|_\infty = \sup_n |x_n|$$

It is known that the α-dual of c_0, c and l_∞ respectively being denoted by c_0^α, c^α and l_∞^α is l_1 which is the space of absolutely convergent series.

In our case, the almost α-dual of c_0, c and l_∞ is $\hat{l}_1$ which is the space of absolutely almost convergent series. This is a natural extension of l_1.

Let D be the shift operator on l_∞, i.e.

$$D(x_n) = x_{n+1}$$

It may be recalled that Banach limit L (see Banach [1]) is a non-negative linear functional on ℓ_∞ such that L is invariant under the shift operator i.e.

$$L(Dx) = L(x)\ \forall x \in \ell_\infty$$

and $L(e) = 1$, where

$$e = \{1,1,1,\dots\}$$

A sequence $x \in \ell_\infty$ is called almost convergent (see Lorentz [4]) if all Banach limits of x coincide. Let $\hat{C}$ denote the set of all almost convergent sequences. Lorentz [4] proved that

$$\hat{C} = \left\{ x\colon \lim_{p\to\infty} \frac{1}{p+1} \sum_{i=0}^{p} x_{n+i} \text{ exists uniformly in } n \right\}$$

A new sequence space $\hat{\ell}_1$ of absolutely almost convergent series, which emerges naturally as an absolute analogue of almost convergence, has been introduced in a natural way by Das, Lalitner and Nanda [2]. Given an infinite series $\sum\ a_n$ being denoted by a, we write

$$x_n = a_0 + a_1 + a_2 + \cdots + a_n$$

Define $d_{p,n}$ as

$$d_{p,n} = d_{p,n}(x) = \frac{1}{p+1} \sum_{i=0}^{p} x_{n+i}\ (p > 0, n \geq 0)$$

By taking

$$d_{0,n} = d_{0,n}(x) = x_{n-1}$$

We then write for $p, n \geq 0$

$$\phi_{p,n} = \phi_{p,n}(a) = d_{p+1,n} - d_{p,n}$$

then

$$\phi_{0,n} = a_n$$

$$\phi_{p,n} = \frac{1}{p(p+1)} \sum_{i=1}^{p} i a_{n+i}\ (p \geq 1)$$

The series a (or the sequence x) is said to be absolutely almost convergent if

$$\sum_{p} \left|\phi_{p,n}\right| \text{ converges uniformly in } n.$$

2 Almost α-dual (Def:)

Using the concept of absolutely almost convergent series, we introduce the concept of almost α-dual of a sequence space of scalars as

$$E^{\hat{\alpha}} = \left\{(a_n) \in S: \sum_{k=1}^{\infty} a_k x_k \text{ absolutely almost converges for } (x_k) \in E\right\}$$

Theorem 2.1: $c_0^{\hat{\alpha}} = c^{\hat{\alpha}} = \ell_\infty^{\hat{\alpha}} = \hat{\ell}_1$
where

$$\hat{\ell}_1 = \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}| < \infty \text{ uniformly for every } n$$

Proof: Since $c_0 \subset c \subset \ell_\infty \Rightarrow \ell_\infty^{\hat{\alpha}} \subset c^{\hat{\alpha}} \subset c_0^{\hat{\alpha}}$. Now we show that $\ell_\infty^{\hat{\alpha}} = \hat{\ell}_1$ and that it is sufficient to show that $c_0^{\hat{\alpha}} \subset \hat{\ell}_1$ so that the theorem is complete.

Let $(x_k) \in \ell_\infty$ and $a = (a_k)$ be a sequence of complex numbers such that

$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}| < \infty \text{ uniformly for every } n$$

Let

$$y_n = \sum_{k=1}^{n} a_k x_k$$

$$d_{p,n} = \frac{1}{p+1} \sum_{i=0}^{p} y_{n+i} \; (p > 0, n \geq 0)$$

$$\phi_{p,n} = \frac{1}{p(p+1)} \sum_{i=1}^{p} i a_{n+i} x_{n+i} \; (p \geq 1)$$

Then

$$\sum_{p=1}^{\infty} |\phi_{p,n}| = \sum_{p=1}^{\infty} \left| \frac{1}{p(p+1)} \sum_{i=1}^{p} i a_{n+i} x_{n+i} \right|$$

$$\leq \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}||x_{n+i}|$$

$$< \sum_{p=1}^{\infty} \sup_{1 \leq i \leq p} |x_{n+i}| \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}|$$

$$< \sup_{1 \leq i < \infty} |x_i| \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}|$$

$$< M \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}|$$

$$< \infty$$

for every n (where $M = \sup_{i \geq 1} |x_i| = \|x\|_\infty$)
Thus

$$\sum_{p=1}^{\infty} |\phi_{p,n}| < \infty, \forall n$$

Converse.

Suppose $a = (a_k)$ be a sequence of complex numbers such that $(a_k) \in \ell_\infty^{\hat{\alpha}}$ but

$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}| = \infty \text{ for some } n$$

Define a sequence $x = (x_k)$, where $x_k = \text{sgn}a_k$, for every k, then $(x_k) \in \ell_\infty$, putting $x_{n+i} = \text{sgn}a_{n+i}$, we obtain

$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} ia_{n+i}x_{n+i} = \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}| = \infty$$

This is a contradiction that $(a_k) \in \ell_\infty^{\hat{\alpha}}$. Now it is sufficient to show that $c_0^{\hat{\alpha}} \subset \hat{\ell}_1$.

Let $a = (a_k)$ be a sequence of complex numbers such that $(a_k) \in c_0^{\hat{\alpha}}$ but $(a_k) \in \hat{\ell}_1$, i.e.

$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}| = \infty \text{ for some } n$$

Since

$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i|a_{n+i}| \leq \sum_{i=1}^{\infty} |a_{n+i}|$$

$$\Rightarrow \sum_{i=1}^{\infty} |a_{n+i}| \text{ diverges for this } n.$$

Define $x = (x_k)$ such that

$$x_k = \begin{cases} 0 & , \forall k \leq n, \\ \text{sgn}\dfrac{a_{n+i}}{i} & , \forall k = n+i, i = 1,2,3,\cdots. \end{cases}$$

Then $(x_k) \in c_0$. Since,

$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} ia_{n+i}x_{n+i} = \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} |a_{n+i}|$$

$$= \sum_{p=1}^{\infty} \left(\frac{1}{p} - \frac{1}{p+1}\right) s_p \left(\text{where } s_p = \sum_{i=1}^{p} |a_{n+i}| \right)$$

Since

$$\sum_{i=1}^{\infty} |a_{n+i}| \text{ diverges, therefore } \sum_{p=1}^{\infty} \frac{s_p}{p}$$

and

$$\sum_{p=1}^{\infty} \frac{s_p}{p+1} \text{ also diverges, therefore the series}$$
$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} i a_{n+i} x_{n+i} = \sum_{p=1}^{\infty} \frac{1}{p(p+1)} \sum_{i=1}^{p} |a_{n+i}|$$

does not converge.
This is a contradiction, therefore $c_0^{\hat{\alpha}} \subset \hat{\ell}_1$, and the proof is complete.

3 Semi Difference sequence space (Def:)

Let w be the space of all complex valued sequences $x = (x_k)$. Kizmaz [3] introduced the difference sequence space

$$\ell_\infty(\Delta) = \{x = (x_k) \in w : (\Delta x_k)_{k=1}^{\infty} \in \ell_\infty\}$$

He further proved that a sequence $(x_k) \in \ell_\infty(\Delta)$ iff
(i)

$$\sup_{k\geq 1} \frac{|x_k|}{k} < \infty$$

(ii)

$$\sup_{k\geq 1} |x_k - k(k+1)^{-1} x_{k+1}| < \infty$$

By considering the first of the above two conditions i.e. (i), we have constructed a new sequence space and call it to be a semi difference sequence space. Thus a semi difference sequence (x_k) is defined as a sequence (x_k) such that

$$\sup_{k\geq 1} \frac{|x_k|}{k} < \infty$$

or a semi difference sequence space

$$\ell_\infty(s\Delta) = \left\{ x = (x_k) \in w : \sup_{k\geq 1} \frac{|x_k|}{k} < \infty \right\} \tag{3.1}$$

Theorem 3.1 : $\ell_\infty(s\Delta)$ is a Banach space with the norm defined by

$$\|x\| = \sup_{k\geq 1} \frac{|x_k|}{k} \tag{3.2}$$

Proof: It is easy to see that $\|x\|$ defined in (3.2) is a well defined on $l_\infty(s\Delta)$. Now we prove that $l_\infty(s\Delta)$ is a complete normed space with respect to the above norm.

Let $\{x^r\}_{r=1}^{\infty}$ be a Cauchy sequence in $l_\infty(s\Delta)$ where $x^r = \{x_k^r\}_{k=1}^{\infty}$ such that

$$\sup_{k\geq 1}\frac{|x_k^r|}{k}<\infty,(r=1,2,3,\cdots)$$

Given $0<\varepsilon<1,\exists$ a positive integer N_0 such that

$$\|x^r-x^s\|<\varepsilon\ \forall r,s\geq N_0$$

or

$$\left|\frac{1}{k}(x_k^r-x_k^s)\right|<\varepsilon,\forall r,s\geq N_0\forall k\geq 1 \quad (3.3)$$

This shows that for a fixed $k(1\leq k<\infty)$, the sequences

$$\left(\frac{x_k^r}{k}\right)_{r=1}^{\infty}$$

is a Cauchy sequence of complex numbers. Since the space of complex numbers is a complete normed space, therefore

$$\left(\frac{x_k^r}{k}\right)_{r=1}^{\infty}$$

is converges in it. Suppose that the limit

$$\lim_{r\to\infty}\frac{x_k^r}{k}=y_k,\ \text{for some } y_k\in C$$

Define $x_k=ky_k$ and $x=(x_k)_{k=1}^{\infty}$, keeping r fixed and taking the limit when $s\to\infty$ in (3.3), we get

$$\left|\frac{1}{k}(x_k^r-x_k)\right|<\varepsilon,\forall r\geq N_0\&\forall k\geq 1$$

or

$$\sup_{k\geq 1}\left|\frac{1}{k}(x_k^r-x_k)\right|\leq\varepsilon,\forall r\geq N_0$$

or

$$x^r\to x\ \text{as } r\to\infty$$

Now we must show that $x\in\ell_\infty(s\Delta)$, we have

$$\begin{aligned}|x|&=\sup_{k\geq 1}\frac{|x_k|}{k}\\&=\sup_{k\geq 1}\frac{|x_k-x_k^r+x_k^r|}{k}\\&<\sup_{k\geq 1}\frac{|x_k-x_k^r|}{k}+\sup_{k\geq 1}\frac{|x_k^r|}{k}\\&<\varepsilon+\ \text{finite number}\\&\Rightarrow x\in\ell_\infty(s\Delta)\end{aligned}$$

Therefore, $\ell_\infty(s\Delta)$ is a Banach space. This completes the proof.

Theorem 3.2 : The α-dual of $l_\infty(s\Delta)$ is $l_\infty^\alpha(s\Delta)$ i.e. D_s,

$$\text{where } D_s = \left\{a = (a_k) \in \omega: \sum_{k=1}^{\infty} k|a_k| < \infty\right\} \tag{3.4}$$

Proof : Let $(a_k) \in D_s$ and $x \in l_\infty(s\Delta)$. Then

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} k|a_k|\frac{|x_k|}{k}$$
$$< \sup_{k\geq 1}\frac{|x_k|}{k}\sum_{k=1}^{\infty} k|a_k|$$
$$< \infty \text{ (using Theorem 3.1\&3.4)}$$

Thus $(a_k) \in l_\infty^\alpha(s\Delta)$.
Conversely, if $(a_k) \in l_\infty^\alpha(s\Delta)$, then

$$\sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for each } x \in l_\infty(s\Delta)$$

Define $x = (x_k)$ where $(x_k) = k$ for each k.

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} |a_k||x_k|$$
$$= \sum_{k=1}^{\infty} k|a_k|$$
$$< \infty \text{ using 3.4}$$
$$\Rightarrow (a_k) \in D_s$$

Therefore,

$$l_\infty^\alpha(s\Delta) = D_s$$

Thus we complete the proof.

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