International Research Journal of Mathematics, Engineering and IT
ISSN: (2349-0322)
Association of Academic Researchers and Faculties (AARF)
Impact Factor- 7.132 Volume 10, Issue 03, March 2023
Website- www.aarf.asia, **Email** : editoraarf@gmail.com

Analyze the correlation between restricted partition function congruences and modular form Fourier coefficients, emphasizing any new number-theoretic implications

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DOI:DOI:18.A003.aarf.J14I01.010184

Abstract: This work aims to examine the occurrence of congruences and thoroughly assess various classes of limited partition functions, particularly in relation to their modular behavior. This study examines the restricted partition functions of the type $p_k(q^2m+b)$ for certain primes q with k=3,5,7. For each k, we look at primes q that satisfy $p_k(q^2m+b) \equiv 0 \bmod k$ for all natural numbers m. One especially productive area of research in partition theory is the study of congruence behavior in constrained partition functions. This is the process of identifying the conditions under which certain functions' values exhibit predictable patterns modulo integers. We then demonstrate our results using a conventional proving technique for each scenario.

Keywords: partition functions, Fourier coefficients**, Congruences,** partition theory, modular functions, restricted partition

1. Introduction:

Mathematicians such as James Joseph Sylvester and Percy MacMahon made further contributions to the development of partitions during the 19th century. They investigated partitions from a combinatorial point of view, and introduced diagrammatic representations, such as Ferrers and Young diagrams, which made it feasible to understand partitions visually as groupings of dots or boxes. Mathematicians were able to classify partitions, uncover novel identities, and investigate partitions with restrictions—including those that involve only odd numbers or different parts—with the use of these tools. It was not until the beginning of the twentieth century that considerable progress was made in the study of partitions, thanks to the work of G. H. Hardy and the Indian mathematical prodigy Srinivasa Ramanujan. Together, they developed new analytic techniques that allowed them to study how the number of partitions grows for very large numbers. Ramanujan also uncovered remarkable and unanticipated congruence qualities in partition numbers, such as the regular behavior they exhibited when divided by particular prime numbers. The significance of partitions was not limited to a simple counting procedure; these findings demonstrated that they were also deeply significant in the field of arithmetic. Later on, Hans Rademacher made improvements to the findings of Hardy and Ramanujan by providing exact formulas for the number of partitions of a number. At this point in time, partitions had become firmly established as a major topic in number theory, acting as a bridge between fields such as combinatorics, analysis, and modular forms. The history of partition theory is therefore a captivating story: it begins with the pioneering ideas of Euler in the eighteenth century, moves on to the developments in combinatorics that occurred in the nineteenth century, and ends with the significant breakthroughs in arithmetic and analysis that took place in the early twentieth century.

A restricted partition is a method of representing a positive integer as a sum of smaller positive numbers, subject to certain precise restrictions or constraints. Restricted partitions differ from the classical, unrestricted partition function in that they set rules for which numbers are allowed to occur in the total and how frequently they are permitted to appear. In contrast, the classical, unrestricted partition function counts all possible sums without regard to the nature of the components. It is feasible to investigate some combinatorial structures and discover numerical patterns that are not observable in the general case with the help of these limits. One form of restriction that is frequently used is one that limits the number of times that a certain component can appear in a partition. For instance, if we take the number 5, its partitions into separate pieces are 5, 4+1, and 3+2. As a result of this constraint, a number of intriguing combinatorial truths emerge. For example, Euler's classical theorem states that the number of partitions into distinct parts is equal to the number of partitions into odd parts Odd portions, which let only odd integers to be included in the partition, are another restriction that has been extensively researched. For the identical example of 5, the partitions into odd parts would be 5, 3+1+1, and 1+1+1+1+1. These kinds of limitations typically uncover unexpected similarities and congruence qualities and play a crucial role in the development of partition theory and modular forms. Partitions that are bounded are those in which the parts must meet particular restrictions, such as being less than or equal to a given number, or appearing no more than a fixed number of times. The partitions of the number 5 with portions not more than 3, for example, would be

3+2, 3+1+1, 2+2+1, and 2+1+1+1. In combinatorics, number theory, and applications in statistical mechanics, bounded partitions are beneficial since constraints are naturally present in counting problems. In general, restricted partitions can be classified according to many different rules, including:

- **Multiplicity restrictions**: limiting how many times a particular number can appear.
- **Parity restrictions**: allowing only even or odd numbers.
- **Arithmetic sequence restrictions**: requiring parts to belong to a specific sequence, such as multiples of a fixed number.
- **Range or bounded restrictions**: restricting parts to lie within a specific interval.

These definitions do not only make it feasible to study new combinatorial identities and congruence relations; they also provide a bridge to deeper mathematical structures, such as modular forms and generating functions. The study of restricted partitions allows mathematicians to explore how constraints influence the arithmetic and combinatorial properties of integers, which in turn leads to more profound theoretical insights and a wide range of applications in modern mathematics.

The foundation of number theory is the identification of fundamental relationships that capture intricate integer properties. Surprisingly, one of the most fundamental mathematics ideas—writing a number as the sum of other numbers—can be used to illustrate the underlying structure of the integers. We refer to this as partition theory. A partition is a representation of a nonnegative number n as the sum of summands, or positive integers, whose order is irrelevant. **Yang (2011)**conducted an investigation into the congruences of the classical partition function and made important contributions to the field of knowledge about the mathematical features of partition numbers. **Folsom, Kent, and Ono (2012)** concentrating on the ℓ-adic characteristics of the partition function and providing an algebraic point of view on congruences. They investigated p-adic expansions of partition numbers and discovered patterns that were compatible with the behavior of modular forms. **Mousavi and Reniers (2005)**an investigation was conducted on structural congruences in formal systems, with a special focus on computational logic and software architectures. Even though their major focus was not on integer partitions, their work demonstrated universal concepts of congruence and predictable behavior in organized sequences. **Maloney and Witten (2010)**Partition functions in the field of quantum gravity in three dimensions were investigated. It was proved through their study that concepts from partition theory, including congruences and modular behavior, could be extended to theoretical physics. It was demonstrated through this work that the modular techniques utilized in combinatorial and number-theoretic contexts might also yield insights into high-level physical phenomena, so bridging the gap between pure mathematics and physics. **Sop and Kozak (2019)** investigated the impact that brand personality, self-congruity, and functional congruity have on consumer loyalty to hotel brands.

Bouckenooghe, Zafar, and Raja (2015) explored the impact of ethical leadership on the job performance of employees, with a particular focus on the mediating effects of goal congruence and psychological capital. The results of their study showed that alignment between the aims of the organization and the objectives of the individual substantially improved the results of the work, emphasizing the practical significance of congruence in the formation of behavior and performance. **Vashisht and S (2017)** investigated the impact that the nature of the game has on the persuasiveness of advertisements in the setting of online gaming. The results of their study shown that a strong alignment between the content of the game and the products being sold substantially improved persuasive effects. **Wang et al. (2019)** performed primary research and meta-analyses on the role that followers play in evaluating leadership behavior within businesses. They discovered that there was a correlation between the consistency of the perceptions of followers and the behaviors of leaders, which affected the correctness of their evaluations and evaluative consistency.

Dyer, Fitzpatrick, and Xin (2018) investigated the limitations that apply to partition functions of conformal field theories (CFT) in two dimensions that are flavored. The partition functions were subjected to modular and symmetry constraints by their research, which facilitated a methodical examination of acceptable physical configurations. This research supported the effectiveness of congruence and structural alignment in the identification of consistent theoretical models and highlighted how modularity controls the behavior of complicated partition functions.The purpose of this study is to analyze the occurrence of Congruences and to carefully evaluate different classes of limited partition functions, especially with regard to the modular behavior of these functions. In doing so, it makes a contribution to a more comprehensive understanding of the ways in which arithmetic, algebraic, and analytic features converge within the theory of modular forms and partitions.

2. Objectives of the study

- **To examine the relationship between modular form Fourier coefficients and limited partition function Congruences, emphasizing any novel number-theoretic implications.**
- **To** analyze the correlation between restricted partition function congruences and modular form Fourier coefficients, emphasizing any new number-theoretic implications

3. Methodology

Theoretical Framework and Mathematical Foundations: A detailed survey of the literature on integer Congruences, modular forms, and partition theory will be the first step in the research. To provide a strong mathematical foundation, important works by Ramanujan, Atkin, Serre, and Ono as well as more recent advancements in the arithmetic of modular forms will be rigorously analyzed. We shall formalize definitions of limited partition functions, including those with odd parts, separate portions, or congruence criteria.

Construction and Analysis of Generating Functions: The next phase of the work will involve creating generating functions for the various limited partition functions that are being investigated. For instance, generating functions for partitions into odd or distinct parts will be represented by endless q-series or eta-products. These generating functions will then be recast using modular or quasi-modular forms as needed. The modular characteristics of these functions, such as level, weight, and transformation behavior, will be analyzed to determine if they exhibit complete or partial modularity.

4. Data Analysis and results

Partition functions are studied in the fields of combinatorics and number theory, with an emphasis on the methods by which integers can be written as the sum of positive integers. Partition theory began as a strictly combinatorial field; nevertheless, the realization that generating functions of partitions frequently display modular or mock modular behavior has established a deep connection to analytic number theory. Modular forms offer a systematic, analytical framework that encodes the arithmetic features of partition sequences within their coefficients, which allows for the rigorous examination of divisibility, symmetry, and congruence patterns. This bridge changes the study of partitions from basic enumeration into a rich subject where viewpoints from combinatorics, arithmetic, and analysis converge. It is now possible to approach issues in partition theory that had previously been unsolvable using purely combinatorial methods by employing modular and analytic techniques through this link. For example, modular forms can be used to methodically study classical congruences established by Ramanujan for unconstrained partitions, as well as expansions of these congruences to restricted or generalized partitions. By utilizing the transformation features and intrinsic symmetries of modular forms, mathematicians are able to identify the underlying principles regulating these congruences, providing insight into the mathematical structure of integers and their distributions. This unification also allows for the construction of new identities, asymptotic formulas, and links between different partition sequences, revealing how modular forms operate as a conduit bridging discrete combinatorial objects to the continuous realm of number theory. We shall examine congruences of $p_3(n)$mod3 in this work. The following derivation involving the generating function of $p_3(n)$ serves as motivation for this, recalling the equation and letting k=3 shows

$$P_3(q) = \sum_{n=0}^{\infty} p_3(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{3n})}{(1-q^n)} \text{.....(1)}$$

We be up to date by that $1 - q^{3n} \equiv (1 - q^n)^3$mod3, as a result

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^n)} \text{ mod3....(2)}$$

This be able of be reduced to

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \left(\prod_{n=1}^{\infty} 1 - q^n\right)^2 \text{ mod3....(3)}$$

Using Euler's Pentagonal quantity Theorem, the equation on top of is transformed into

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}\right)^2 \text{ mod3....(4)}$$

This can now be articulated as a twice sum

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \sum_{k,\ell=-\infty}^{\infty} (-1)^{k+\ell} q^{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}} \text{mod3....(5)}$$

at what occasion

$$n = \frac{k(3k-1)}{2} + \frac{\ell(3\ell-1)}{2},$$

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \sum_{n=0}^{\infty} q^n \sum_{k,\ell} (-1)^{k+\ell} \text{ mod3.....(6)}$$

$$p_3(n) \equiv \sum_{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}=n} (-1)^{k+\ell} \text{mod3 .... (7)}$$

$$n = \frac{k(3k-1)}{2} + \frac{\ell(3\ell-1)}{2} \text{...(8)}$$

$$24n + 2 = (6k-1)^2 + (6\ell-1)^2 \text{ ....(9)}$$

The manipulation shows that when n is expressed as the sum of two pentagonals, it may be seen as a sum of squares problem. This is useful since the problem of representing an integer as a sum of squares has been thoroughly studied. In order to simplify the remainder of the paper, we'll let

$$\kappa = (6k-1) \text{ and } \lambda = (6\ell - 1).$$

$$24n + 2 = (6k-1)^2 + (6\ell-1)^2 = \kappa^2 + \lambda^2 \ldots.(10)$$

Let n be of the form $n = q^2m + b$, where q is a prime number. Every prime, q, is equal to one or three modulo four if the number 2 is taken out of the equation. Let's start by looking at primes q≡3mod4.

Theorem 1: $p_3(q^2m + b) \equiv 0\text{mod}3$, for each and all one natural information m, somewhere q is a prime superior than 3 on behalf of which $q \equiv 3\text{mod}4$ with b flattering $q||24b + 2$.

Proof: We identify that $n = q^2m + b$, so call to mind equation 10.

$$24n + 2 = 24q^2m + 24b + 2 = \kappa^2 + \lambda^2 \text{ ....(11)}$$

In outlook of the fact that $q||24b + 2$, we be up to date with that we can thing out exactly solitary q to obtain

$$q\left(24qm + \frac{24b+2}{q}\right) = \kappa^2 + \lambda^2 = (6k-1)^2 + (6\ell-1)^2 \ldots.(12)$$

This let be acquainted with us that there stay alive no (κ, λ), as well as on behalf of this reason no (k, ℓ), that influence equation (12).

$$p_3(n) \equiv \sum_{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}=n} (-1)^{k+\ell}\text{mod}3 \text{….(13)}$$

$$p_3(q^2m + b) \equiv 0 \text{ mod}3 \text{….(14)}$$

After describing the congruences for primes congruent to three modulo four, we need to look into primes congruent to one modulo four. This will be much more challenging because we cannot simply apply Theorem 2. As it happens, these primes need to be considered modulo 12. For each prime q=1mod4, we know that q=1,5mod12. Numerical data makes it easy to see that when q."1mod12, no congruence features are visible. Therefore, we want to investigate the scenario where q≥5mod12. We will apply our methods to all primes q≥5mod12 after establishing a certain congruence.

Theorem 2: $p_3(5^2m + b) \equiv 0\text{mod}3$, in maintain of all ordinary numbers m, by b pleasurable $5||24b + 2$.

Proof: We plac$e\ of\ protection = 5^2m + b$. we distinguish that

$$24n + 2 = 600m + 24b + 2 = 5\left(120m + \frac{24b+2}{5}\right) = \kappa^2 + \lambda^2, \text{….(15)}$$

everyplace $\frac{24b+2}{5} \in \mathbb{Z}$ as $5 \| 24b + 2$.

This acquaint by us that 5 divides $\kappa^2 + \lambda^2$. Regrettably important that a figure separate a sum be not awfully prepared to lend a hand, so we aspiration for to inscribe $\kappa^2 + \lambda^2$ as a creation. In support of this we think the Gaussian Integers in adding to write $\kappa^2 + \lambda^2 = (\kappa + \iota\lambda)(\kappa - \iota\lambda)$. So $5 \mid (\kappa + \iota\lambda)(\kappa - \iota\lambda)$. We are familiar with on the additional hand that $5 \nmid (\kappa + \iota\lambda)$ in addition to $5 \nmid (\kappa - \iota\lambda)$ since if five part one it divides the extra because they are versatile conjugates. This forces $5^2 \mid (\kappa + \iota\lambda)(\kappa - \iota\lambda)$, which say the opposite to our outward appearance that $5 \| 24b + 2$. It is broadly known that five can be written as a creation of two primes in the Gaussians, namely$5 = (2 + \iota)(2 - \iota)$.

Thus we differentiate that $(2 + \iota)(2 - \iota) \mid (\kappa + \iota\lambda)(\kappa - \iota\lambda)$.

Now take for granted that $(2 + \iota) \mid (\kappa + \iota\lambda)$.

Subsequently

$$(\kappa + \iota\lambda) = (2 + \iota)(\kappa' + \iota\lambda') \ldots.(16)$$

The real denominator willpower not involves the conjugation so we come across:

$$\begin{aligned} \kappa^* + \iota\lambda^* &= [(6k - 1) + \iota(6\ell - 1)][3 - 4\iota] \\ &= [18k + 24\ell - 7] + \iota[18\ell - 24k + 1] \qquad \ldots(17) \\ &= [6(3k + 4\ell - 1) - 1] + \iota[6(-3\ell + 4k) - 1]. \end{aligned}$$

As a result

$$k^* = 3k + 4\ell - 1 \text{ and } \ell^* = -3\ell + 4k \ldots.(18)$$

For that motive

$$(-1)^{k^*+\ell^*} = (-1)^{7k+\ell-1} = (-1)^{k+\ell-1} = -[(-1)^{k+\ell}] \ldots(19)$$

In sight of the fact that

$$p_3(n) \equiv \sum_{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}=n} (-1)^{k+\ell} \text{mod} 3 \ldots.(20)$$

$$p_3(5^2m + b) \equiv 0 \text{ mod} 3 \ldots..(21)$$

It now appears probable to generalize this way to other primes, $q \equiv 5\text{mod}12$, besides 5 . This can be complete but first we require to establish a Lemma.

5. Conclusion

Congruences for constrained partition functions, which are frequently demonstrated using Ramanujan's theta-function identities or linkages to other combinatorial identities like Euler's Pentagonal Number Theorem, reveal divisibility features and patterns in partitions with limitations. Similar to the traditional Ramanujan congruences for the unrestricted partition function p(n), researchers examine these congruences to discover infinite families of divisibility results for a variety of restricted functions. These discoveries advance our knowledge of the arithmetic characteristics of partitions.

6. Limitations

Similar to Ramanujan's well-known congruences for the unrestricted partition function p(n), congruences for restricted partition functions show similarities in the behavior of these functions modulo specific numbers. The word "limitation" refers to the difficulties and complexity involved in locating such congruences for restricted cases, especially the lack of general Ramanujan-type congruences for many restricted functions and the complexity of the restrictions themselves, which frequently necessitates specialized, case-by-case analysis.

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