International Research Journal of Natural and Applied Sciences
ISSN: (2349-4077)
Impact Factor 7.032 Volume 10, Issue 07, July 2023
Website- www.aarf.asia, **Email** : editor@aarf.asia , editoraarf@gmail.com

A Variation of non-additive Functions on a Fuzzy σ-Algebra

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Abstract

The present paper aims at introducing the concept of orthogonality of functions in a Fuzzy σ-algebra *μ*; which is the appropriate generalization of the concept of disjointness in theory of fuzzy sets. The notions of null-additivity, null-null-additivity and *σ*-null-additivity, using the concept of orthogonality, are introduced and studied. In this paper, we have established the existence of three kinds of variations, namely orthogonal variation $|m|_0$, chain variation $|m|_c$ and inclusion variation $|m|_i$ of a functions *m* defined on fuzzy *σ*-algebra *μ* and it is also shown that, in general, they are not unique. We also studied the properties as null additivity, null-null-additivity, exhaustively and so on and relations between these three kinds of variations on *μ*. A Jordan type decomposition theorem for an extended real-valued function, which is of bounded chain variation defined on *μ* is established. Also, a Jordan type decomposition theorem for real-valued function *m* defined on *μ*, where the decomposed parts involve the positive and negative inclusion variations m_i^+ and m_i^- of *m*, is established. Finally, introducing the concept of *m*-atom of a real-valued function *m* on *μ*, it is shown that (for a suitable *m*) *m* is non-atomic if and only if its inclusion variation $|m|_i$ is non-atomic.

2.Preliminaries

2.1 A fuzzy set in a nonempty set *X* is an element of the family of I^X of all functions from *X* to the closed unit interval $I = [0,1]$ of the real line. A fuzzy set which assigns to each *x* in *X* the value *t*, $t \in I$, is denoted by *t*. A sequence $\{\lambda_i\}_{i=1}^{\infty}$ of fuzzy sets in *X* increases to $\lambda \in I^X$ (written as $\lambda_i \uparrow \lambda$) if $\{\lambda_i(x)\}_{i=1}^{\infty}$ is monotonic increasing and converges to $\lambda(x)$ for each *x* in *X*.

2.2 A fuzzy *σ*-algebra *μ* on a nonempty set *X* is a subfamily I^X of satisfying:

A1. $1 \in \mu$

A2. $\lambda \in \mu \Rightarrow 1-\lambda \in \mu$

A3. if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in *μ*, then $\overset{\infty}{\underset{i=1}{\vee}} \lambda_i = \sup \lambda_i \in \mu$.

If N_1 and N_2 are fuzzy *σ*-algebra on *X*, then $N_1 \vee N_2$ denotes the smallest fuzzy *σ*-algebra on *X*, containing $N_1 \vee N_2$. Arbitrary intersection of fuzzy *σ*-algebra on a set *X*, is a fuzzy *σ*-algebra on *X*. For $S \subseteq I^X$, [*S*] denotes the smallest fuzzy *σ*-algebra containing *S* [ ].

2.3.[ ] Let *X* be a nonempty set.

(i) [{0}] = {0, 1} is the trivial fuzzy algebra on *X*.

(ii) For $\lambda \in I^X, [\{\lambda\}] = \{\lambda, \lambda', \lambda \vee \lambda', \lambda \wedge \lambda', 0, 1\}$.

(iii) Let (X, B, P) be a classical probability measure space. Then $\hat{B} = \{\chi_A : A \in B\}$ is a fuzzy *σ*-algebra on *X*.

Note that if $\lambda_i \in \mu$, then by [ ], we have $\overset{\infty}{\underset{i=1}{\wedge}} \lambda_i \in \mu$.

3. **Null Additive function**

Definitions 3.1 Let *m* be an extended real valued functions defined on μ i.e. $m : \mu \to [-\infty, \infty]$. We say that

(i) *m* monotone if $f, g \in \mu$ and $f \le g$ implies $m(f) \le m(g)$.

(ii) *m* is continuous from below, if , $\{f_n\} \in \mu$ and $f_1 \le f_2 \le \ldots$ imply

$$m(\overset{\infty}{\underset{n=1}{\vee}} f_n) = \lim_{n\to\infty} (f_n).$$

(iii) *m* is continuous from above, if $\{f_n\} \in \mu$ and $f_1 \ge f_2 \ge \ldots$ with $|m(f_n)| < \infty, n \ge 1$, imply $m(\overset{\infty}{\underset{n=1}{\wedge}} f_n) = \lim_{n\to\infty} (f_n)$.

Definition3.2 The functions *f* and *g*, $f, g \in \mu$ are said to be orthogonal, denoted by $f \perp g$ if and only if $m(f \wedge g) = 0$. The sequence $\{f_i\}$ of functions from μ are said to be mutually orthogonal if and only if $m(f_i \wedge f_j) = 0$ for $i \ne j$.

Definition 3.3 (i) A function $m : \mu \to [-\infty, \infty]$] is called null- null additive, if $m(f \vee g) = 0$ whenever $f, g \in \mu, f \perp g$, and $m(g) = 0$.

Example 3.1. (i) Let $m(f) \ne 0$, whenever $f \in \mu$ and $f \ne 0$. Then *m* is a null-additive function.

(ii) Let $X = \{x, y\}$ and $\mu = \{0, \chi_{\{x\}}, \chi_{\{y\}}, 1\}$. Define $m : \mu \to [0, \infty]$ by $m(1) = 1$ and $m(f) = 0, f \ne 1 \in \mu$. Then *m* not a null-additive function.

(iii) Let $m : \mu \to [0,1]$ be a function defined by $m(f) = \{^{2\mu(f), \mu(f) \le 1/2.}_{-2\mu(f)+2, \mu(f) \ge 1/2}, f \in \mu$, where $m : \mu \to [0,1]$ is a function satisfying $\mu(f \vee g) = \mu(f) + \mu(g)$ for $f, g \in \mu$ and $f \perp g$ .Then *m* is null-additive function.

Definition 3.4 A function $m : \mu \to [0, \infty]$ with $m(f) = 0$ is called a *σ*-null-additive if for every $f \in \mu$ and for any sequence $\{g_i\}$ of function from μ such that $f \perp g_i, i \in N$, and $m(g_i) = 0$, we have $m(f \vee \overset{\infty}{\underset{i=1}{\vee}} g_i) = m(f)$.

Theorem 3.1 Let $m : \mu \to [0, \infty]$ be a monotone, null-additive and continuous from above function. Then $\lim_{n\to\infty} m(f \vee g_n) = m(f)$ for $f \in \mu$ and for any decreasing sequence $\{g_n\}$ of functions from μ for which $\lim_{n\to\infty} m(g_n) = 0$ and there exists at least one n_0 such that $m(f \vee g_{n_0}) < \infty$ as $m(f) < \infty$.

Proof. Let $g = \wedge_{n=1}^{\infty} g_n$. Then $m(g) = m(\wedge_{n=1}^{\infty} g_n) = \lim_{n\to\infty} m(g_n)$. Also, by [ ], we have $\wedge_{n=1}^{\infty} (f \vee g_n) = f \vee g$. Hence from $(f \vee g_n) \downarrow (f \vee g)$, and null-additivity and continuity from above of *m*, we get $\lim_{n\to\infty} m(f \vee g_n) = m(f \vee g) = m(f)$.

Proposition 3.1. Let $m : \mu \to [0, \infty]$ be a null-additive function. Then

(i) *m* is null-additive.

(ii) $m(g_i) = 0$ for all $i \in N$ implies $m(\overset{\infty}{\underset{i=1}{\vee}} g_i) = 0$ for a sequence $\{g_i\}$ of mutually orthogonal functions from μ.

Proof. (i) Let $m(g) = 0$ and $f \perp g,\ f, g \in \mu$. Then taking $g_1 = 0$ and for $g_i \geq 2$, we have by definition of σ-null-additivity that

$$m(f \vee g) = m(f \vee \overset{\infty}{\underset{i=1}{\vee}} g_i) = m(f).$$

which yields that m is null-additive.

(ii)) By σ-null-additivity of m, we have $\{f_n\} \in \mu$

Proposition 3.2. Let $m : \mu \to [0, \infty]$ be a monotone, null-additive and continuous from below. Then m is a σ-null additive function.

Proof. Let $f \in \mu, \{g_i\}$ be a sequence of nutually orthogonal functions from μ such that $f \perp g_i$ and $m(g_i) = 0$. Let $h_n = \overset{n}{\underset{i=1}{\vee}} (f \wedge g_i)$. The $\{h_n\}$ is increasing, and $\underset{n \to \infty}{Lim}\, m(h_n) = m\left(\overset{\infty}{\underset{i=1}{\vee}} (f \wedge g_i) \right) = m\left(f \wedge \left(\overset{\infty}{\underset{i=1}{\vee}} \right) g_i \right)$. Since m is monotone and null additive, by principle of induction, we get $\underset{n \to \infty}{Lim}\, m(h_n) = 0\ n \in N$, which yields that m is a σ-null-additive function.

Proposition 3.4. Let $m : \mu \to [0, \infty]$ be a function satisfying $m(f - g) = m(f) - m(g),\ f, g \in \mu$. Then m is null-additive if and only if $m((f \vee g) - (f \wedge g)) = m(f)$, whenever $f \perp g$ and $m(g) = 0$.

4. Existence of variation and a Jordan type decomposition theorem.

Definition 4.1. Let m be an extended real-valued function defined on μ, i.e. $m : \mu \to [-\infty, \infty]$, with $m(0) = 0$. Then variation of m is a function $\eta : \mu \to [0, \infty]$ with the following properties :

(i) for every $f \in \mu$, we have $0 \leq \eta(f) \leq \infty$;

(ii) $\eta(0) = 0$;

(iii) $|m(f)| \leq \eta(f),\ f \in \mu$;

(iv) η is monotone;

(v) $\eta(f) = 0$ if and only if m(g) = 0 for every $g \leq f,\ f, g \in \mu$. Obviously for every $f \in \mu,\ \eta(f) \geq \sup\{|m(g)| : g \leq f,\ g \in \mu\}$ and for $g \leq f,\ f, g \in \mu$, we obtain by (iii) and (iv), $\eta(f) \geq \eta(g) \geq |m(g)|$.

Definition 4.2. Let m be an extended real-valued function defined on μ, with $m(0) = 0$. For every $f \in \mu$, we define

$$|m|_0(f) = \sum_{i-1}^{n} |m(f_i)|,$$

$$m_0^+(f) = \sup \sum_{i-1}^{n} \max\{m(f_i), 0\}$$

$$m_0^-(f) = \sup \sum_{i-1}^{n} \max\{-m(f_i), 0\}$$

where the supremums are taken over all finite families $\{f_i\}$ of mutually orthogonal functions from μ such that $fi \leq f\ (1 \leq i \leq n)$. We say $|m|_0,\ m_0^-$ and m_0^- the orthogonal variation, positive orthogonal variation and negative orthogonal variation, respectively. We say that m is of bounded orthogonal variation (on μ) if $|m|_0(1)<\infty$. it is evident that $|m|_0(1)<\infty$ if and only if $m_0^+(1)<\infty$ and $m_0^-(1)<\infty$.

Note that for $m:\mu\to R(or\ [0,\infty])$ with $m:(0)=0\ (i=1,2)$ and $C\in R$, we have $|m_1+m_2|_0\le|m_1|_0+|m_2|_0$ and $c|m_1|_0=|m_1|_0$. Also $sup\{|m(g)|:g\le f,\ g\in\mu\}\le|m_0|(f),\ f\in\mu.$

Definition 4.3. Let m be a real-valued function defined on μ, i.e. $m:\mu\to(-\infty,\infty)$, with $m(0)=0$. For every $a\in\mu$, we define

$$|m|_c(f)=\sup\sum_{i-1}^{n}|m(f_i)-m(f_{i-1})|,$$

$$m_0^+(f)=\sup\sum_{i-1}^{n}\max\{m(f_i)-m(f_{i-1}),\ 0\},$$

$$m_c^-(f)=\sup\sum_{i-1}^{n}\max\{-(m(f_i)-m(f_{i-1})),\ 0\},$$

where the supremums are taken over all finite chains $\{f_i\}$ in μ between 0 and f, i.e., $0=f_0\le f_i\le.....\le f_n=f.\ f_i\in\mu,1,2,...,n.$ We call $|m|_c$, m_c^+ and m_c^-, the chain variation, positive chain variation and negative chain variation ($m\mu$), respectively. We say that m is of bounded chain variation, if $|m|(1)<\infty$. It is evident that $|m|(1)<\infty$ if and only if $m^+(1)<\infty$ and $m^-(1)<\infty$. Note that $\sup\{m(g):g\le f,g\in\mu\}\le|m|_c(f),\ f\in\mu$ and for a [0,∞] valued monotone function m, with $m(0)=0$, $|m|_c(f)=m(f)\ for\ \ f\in\mu.$

Definition4.4. Let m be an extended real –valued function defined on μ, with $m(0)=0$. For every $f\in\mu$ we define

$$m_i^+(f)=\sup\{m(g):g\le f,g\in\mu\},$$

$$m_i^-(f)=\sup\{-m(g):g\le f,g\in\mu\},$$

$$|m|_i(f)=m_i^+(f)+m_i^-(f).$$

We say that m_i^+, m_i^- and $|m|_i$ the positive inclusion variation, negative inclusion ,variation and inclusion variation of m (on µ),respectively. We say that m is of bounded inclusion variation if $|m|(i)<\infty$. Also $\|m|_i|_i(f)=|m|_i(f),f\in\mu.$ Note that $\sup\{m(g):g\le f,g\in\mu\}\le|m|_c(f),\ f\in\mu$ and for a [0, ∞] valued monotone function m, with m(0) = 0, $|m|_c(f)=m(f)\ for\ \ f\in\mu.$

Theorem4.1.(The existence of variation).For every extended real valued defined on μ,with $m(0)=0$, always exists its variation, which is not unique.

Proof.We have to show:

If m be an extended real valued defined on μ,with $m(0)=0$.Then orthogonal variation $|m|_0$ of m has the following properties:

(i) $0\le|m|_0(f)\le\infty(f\in\mu;$

(ii) $|m|_0(0)=0;$

(iii) $|m(f)|\le|m|_0(f)\,f\in;\mu$

(iv) $|m|_0(f)$ is monotone;

(v) $|m|_0(f)=0$ if and only if m(g)=0 for every $g\le f,g\in\mu;$

(2) Let m be an extended real valued function defined on μ,with m(0)=0. Then the chain variation $|m|_c$ of m has the following properties for $f \in \mu$:

(i) $0 \le |m|_c(f) \le \infty$;

(ii) $|m|_c(0) = 0$;

(iii) $|m(f)| \le |m|_c(f)$;

(iv) $|m|_c(f)$ is monotone;

(v) if $|m|_c(f) = 0$ and only if *m*(*g*)=0 for every $g \le f, g \in \mu$.

(3) Let m real-valued defined on μ,with *m*(0)=0. Then the inclusion variation $|m|_i$ of *m* has the following properties for $f \in \mu$:

(i) $0 \le |m|_i(f) \le \infty$;

(ii) $|m|_i(0) = 0$;

(iii) $|m(f)| \le |m|_i(f)$;

(iv) $|m|_i(f)$ is monotone;

(v) $|m|_i(f) = 0$ if and only if m(g)=0 for every $g \le f, g \in \mu$.

The basic idea for the proofs of the above results is taken from [ ] and [,]. The proofs are immediate, therefore leaved. Finally, it remains to show $|m|_0 = |m|_c = |m|_i$. For this, let $X = \{x, y\}$ and $\mu = \{0, \chi_{\{x\}}, \chi_{\{y\}}, 1\}$ .Define $m : \mu \to R$ by $m(0) = 0, m(\chi_{\{x\}}) = 2, m(\chi_{(y)}) = 3$ and $\mathrm{m}(1) = -8.$. Then $|m|_0(1) = 5, |m|_i = 8,$ and $|m|_i(2) = 11.$

Definition4.5. Afunction $m : \mu \to R$ is called superadditive, if for any family $\{f_i\}_{i \in I}$ of mutually orthogonal functions from μ,we have

$$\sum_{i \in I} m(f_i) \le m(\vee_{i \in I} f_i) .$$

Theorem 4.2. Let $m : \mu \to [0, \infty]$ be a monotone function, with *m*(0)=0. Then its orthogonal variation $|m|_0$ is superadditive.

Proof. Let us suppose that m $\neq$ 0. Let $g_1, g_2 \in \mu$ such that $g_1 \perp g_2$ and $m(g_i) \neq 0$ $(i = 1 or\, i = 2)$. Take an arbitrary but fixed $r \in R$ such that $|m| < (g_1) + |m|_0(g_2) > r.$ Then there exists two numbers $r_1 \in R$ and $r_2 \in R$ such that $r = r_1 + r_2$ and $|m|_0(g_1) > r_1, |m|_0(g_2) > r_2.$ Thus there exists two functions $\{f_i^1\}_{1 \le i \le n}$ and $\{f_j^2\}_{i \le j \le k}$ of mutually orthogonal functions from μ with $f_i^1 \le g_1 (i = 1,2.......,n)$ $f_j^2 \le g_2 (j = 1,2,......,n)$ such that $\sum_{i=1}^{n} m(f_i^1) > r_1$ and $\sum_{i=1}^{k} m(f_j^2) > r_2$,respectively.

The family $\{f_1^1, f_2^1, ..., f_n^{1,}, f_1^1, f_2^2, ..., f_k^2\}$ consists of mutually orthogonal functions from μ with $f_i^1, f_j^2 \le g_1 \vee g_2 (i = 1,2,...,n, j = 1,2,...,k)$ and we obtain $|m|_0(g_1 \vee g_2) \ge \sum_{i=1}^{n} m(f_i^1) + \sum_{j=1}^{k} m(f_j^2).$

Therefore by the inequalities (1) and (2), we obtain that,

$$|m|_0(g_1 \vee g_2) > r_1 + r_2 = r. \tag{3}$$

Further the inequalities (1) and (3) yields that $|m|_0(g_1 \vee g_2) \ge |m|_o (g_1) + |m|_o (g_2).$ Using principle of induction, we can show that the preceeding inequality holds for every finite family $\{f_i\}_{i \in I}$ of mutually orthogonal functions from μ, i.e, $|m|_0(\vee_{i\in J} f_i) \geq \sum_{i\in J} |m|_o(f_i)$. For an arbitrary family $\{f_i\}_{i\in I}$ of mutually orthogonal functions from μ, the preceeding step yields for each finite subset J of I,

$$|m|_o(\vee_{i\in J} f_i) \geq \sum_{i\in J} |m|_o(f_i)$$

which yields that $|m|_0(\vee_{i\in J} f_i) \geq \sum_{i\in I} |m|_0(f_i)$, i.e, $|m|_0$ is superadditive.

Proposition 4.1. Let $m:\mu \to [0,\infty]$ be a superadditive function. Then for every $f \in \mu$, we have

(i) $|m|_0(f) = \sup\{m(g) : g \leq f, g \in \mu\}$

(ii) $|m|_0(f) \geq \sup\{|m|_0(f \wedge g) : g \in \mu\}$.

Moreover, if *m* is monotone then

(iii) $|m|_0(f) = m(f)$.

Proof. (i) Let $\{f_i\}_{1\leq i\leq n}$ be a finite family of mutually orthogonal functions from μ such that $f_i \leq f$. Let $h = \vee_{i=1}^{n} f_i$. Then, by superadditivity of *m*, we assertion.

(ii)obvious.

(iii)Monotonocity of *m* and (i) implies that $m(f) \leq |m|_0(f) = \sup\{m(g) : g \leq f, g \in \mu\} \leq m(f)$.

Propsition 4.2 Let $m:\mu \to [0,\infty)$ be a function. Then $\big||m|_0\big|_0 = |m|_{0.}$

Proof. Let $f \in \mu$. Since for each finite family $\{f_i\}_{1\leq i\leq n}$ of mutually orthogonal functions from μ such that $f_i \leq f$, we have

$$\sum_{i=1}^{n} |m(f_i) \leq |\sum_{i=1}^{n} |m|_0(f_i) \leq \big\||m|_0\big|_0(f)$$

which yields that $|m|_0(f) \leq \big\||m|_0\big|_0(f)$. Further monotonicity of $|m|_0$ and proposition 4.1 (i) and (ii), yields that $|m|_0(f) \geq \sup\{|m|_0(g) : g \leq f, g \in \mu\} = \big||m|_0\big|_0(f)$.

Proposition4.3. Let $m:\mu \to [0,\infty]$ be a monotone and null-additive function with *m*(0)=0. Then its orthogonal variation $|m|_0$ is also null-additive.

Proof. Let $g \in \mu$ such that $|m|_0(g) = 0$. Then, by Theorem 4.1 (i) and (iii), *m*(g) = 0. For an arbitrary $f \in \mu$ such that $f \perp g$, we have

$$|m|_0(f \vee g) = \sup\{\sum_{i-1}^{n} |m(f_i)| : \{f_i\}'s \text{ are mutually orthogonal functions from } \mu$$
$$\text{such that } f_i \leq f \vee g(1 \leq i \leq n)\}.$$
$$= \sup\{\sum_{i=1}^{n} m(f_i \wedge f) \vee (f_i \wedge g) : \{f_i\}'s \text{ are mutually orthogonal functions}$$
$$\text{from } \mu \text{ such that } f_i \leq f \vee g(1 \leq i \leq n)\}.$$

Since *m* is null-additive, we get

$$|m|_0(f \vee g) = \sup\{\sum_{i-1}^{n} |m(f_i \wedge f)| : \{f_i\}'s \text{ are mutually orthogonal functions from } \mu$$
$$\text{such that } f_i \leq f \vee g(1 \leq i \leq n)\}.$$

$=\sup\{\sum_{i=1}^{n}|m(h_i)| : \{h_i\}$'s are mutually orthogonal functions

from μ such that $h_i \le f(1 \le i \le n)\}$.

$=|m|_0(f)$.

Proposition 4.4.Let $m:\mu \to [0,\infty]$ be a monotone function with, $m(0)=0$.Then $|m|_c(f)=m(f), f \in \mu.$ Moreover, if m is superadditive, then $|m|_c(f)=|m|_0(f)=m(f), f \in \mu.$

Proof. The first part of the proposition is immediate of by definition of monotonicity of *m*. For the second part of the proposition, by proposition 4.1 (i), we have

$$|m|_0(f) == \sup\{m(g) : g \le f, g \in \mu\}.$$

Further, since m is monotone, we have $|m|_0(f)=m(f)$. Hence $|m|_c(f)=|m|_0(f)=m(f)$.

Theorem 4.3 Let $m:\mu \to [0,\infty]$ be a monotone function with $m(0) = 0$. Then its orthogonal variation $|m|_0$ is the smallest variation of all superadditive variation of *m*.

Proof. By Theorem 4.2, $|m|_0$ is superadditive. For an arbitrary finite family $\{f_i\}_{i\in I}$ of mutually orthogonal function from μ such that $f_i \le f(i \in I)$ and any superadditive variation η of *m*, we have

$$\eta(f) \ge \eta(\vee_{i\in I} f_i) \ge \sum_{i\in I} \eta(f_i) \ge \sum_{i\in I} |m(f_i)|.$$

Hence $\eta(f) \ge |m|_0(f)$.

Proposition 4.5. If *m* be an extended real valued defined on μ,with $m(0) = 0$. Then the positive and negative orthogonal variation of m, m_o^+ and m_o^- have the following properties:

(i) $0 \le m_o^+(f) \le \infty, 0 \le m_o^-(f) \le \infty (f \in \mu)$;

(ii) $m_0^+(0) = m_0^-(0) = 0$;

(iii) m_o^+ and m_o^- are monotone;

(iv) $m_0^+(0) = (-m)_0^-, m_0^- = (-m)_0^+$;

(v) $|m|_0(f) \le m_0^+(f) + m_0^-(f),\ |m|_0(f) \ge m_0^+(f) \vee m_0^-(f)\ (f \in \mu)$;

(vi) $|m|_0(f) = 0$ if and only if $m_0^+(f) = m_0^-(f) = 0 (f \in \mu)$;

(vii) $m_0^+(f) \ge m(f) \ge -m_0^-(f)(f \in \mu)$.

Proof. The proof of (i) and (vii) are immediate concequences of definition.

Theorem 4.5. Let *m* be an extended real-valued function defined on μ, with $m(0) = 0$. If there exist non negative, superadditive functions γ_1 and γ_2 defined on μ such that $m = \gamma_1 - \gamma_2$, then $\gamma_1 \ge m_0^+$ and $\gamma_2 \ge m_0^-$ where m_o^+ and m_o^- are the positive and negative orthogonal variation of *m*, respectively.

Proof. Let $f \in \mu$ and $\{g_i\}_{1\le i\le n}$ be a finite family of mutually orthogonal functions from μ such that $g_i \le f$. Put $g = \vee_{i=1}^{n} g_i$. Then $g \le f, g \in \mu$. By the inequalities $m \le \gamma_1, -m \le \gamma_2$ non- negativity monotonicily of γ_1 and γ_2 , we have

$$\sum_{i=1}^{n} \max\{m(g_i), 0\} = \sum^{+} m(g_i) \le \sum^{+} \gamma_1(g_i) \le \sum_{i=1}^{n} \gamma_1(g_i) \le \gamma_1(g) \le \gamma_1(f),$$

$$\sum_{i=1}^{n} \max\{-m(g_i), 0\} = \sum^{-} - m(g_i) \le \sum^{+} \gamma_2(g_i) \le \sum_{i=1}^{n} \gamma_2(g_i) \le \gamma_2(g) \le \gamma_2(f),$$

where $\sum^{+}(\sum^{-})$ is taken over those i for which $m(g_i) \geq 0 (m(g_i) < 0)$. Hence $m_0^{+}(f) \leq \gamma_1(f)$ and $m_0^{-}(f) \leq \gamma_2(f)$.

Proposition4.6. Let m be an extended real-valued function defined on μ, with $m(0) = 0$. If m is null-null-additive, then its orthogonal variation $|m|_0$ is also null-null-additive.

Proof. Let $f, g \in \mu, f \perp g$ and $|m|_0(f) = |m|_0(g) = 0$, then by Theorem 4.1(i) (v),we have $m(f^1) = 0$ for $f^1 \leq f, f^1 \in \mu$ and $m(g^1) = 0$ for $g^1 \leq g, g^1 \in \mu$. Thus for an arbitrary $h \leq f \vee g, h \in \mu$, by null-null-additivity of m, we have $m(h) = m(h \wedge (f \vee g)) = m((h \wedge f) \vee (h \wedge g)) = 0$ and it follows that $|m|_0(f \vee g) = 0$.

Proposition 4.7. If non-negative an extended real-valued function defined on μ, with $m(0) = 0$ is continuous from below, then its orthogonal variation $|m|_0$ is also continuous from below.

Proof. Let $\{f_n\}$ be an increasing sequence of functions from μ. Let $\{g_i\}_{1 \leq i \leq k} (k \in N)$ be an arbitrary finite family of mutually orthogonal functions from μ and such that $g_i \leq \bigvee_{n=1}^{\infty} f_n (1 \leq i \leq k)$. Then $g_i \wedge f_n \uparrow g_i$ as $n \to \infty$ for each fixed i $(1 \leq i \leq k)$ and, since m is continuous from below, by the definition of $|m|_0(f_n)$ we have

$$\sum_{i=1}^{k} |m(g_i)| = \sum_{i=1}^{k} |\lim_{n\to\infty} m(g_i \wedge f_n)|$$

$$= \lim_{n\to\infty} \sum_{i=1}^{k} |m(g_i \wedge f_n)|$$

$$\leq \lim_{n\to\infty} |m|_0(f_n).$$

Since $\{g_i\}_{1 \leq i \leq k}$ is an arbitrary, finite family of mutually orthogonal functions from μ and $g_i \leq \vee_{n=1}^{\infty} f_n (1 \leq i \leq k)$, we get by the last inequality

$$|m|_0(\bigvee_{n=1}^{\infty} f_n) \leq \lim_{n\to\infty} |m|_0(f_n).$$

Together with the opposite inequality from the monotonicity of $|m|_0$, we obtain the desired conclusion.

Definition4.6. A function $m : \mu \to [-\infty, \infty]$ is said to be exhaustive, if $\{f_n\} \in \mu$ such that $f_i \perp f_j i \neq j$ implies $\lim_{n\to\infty} m(f_n) = 0$.

Proposition 4.8. Let m be a non-negative extended real-valued function defined on μ. Then its orthogonal variation $|m|_0$ is exhaustive and m itself is also exhaustive.

Proof. Let $|m|_0 (1) < \infty$. On the contrary, let $|m|_0$ is not exhaustive, then there would exist an $\varepsilon_0 > 0$ and a sequence $\{f_n\}$ of mutually orthogonal functions from μ such that

$$|m|_0 (f_n) \geq \varepsilon_0 \text{ for } n = 1,2,\ldots \tag{1}$$

On the otherhand, there exists a finite family $\{g_i^n\}_{1 \leq i \leq k_n}$ (for each $n \geq 1$) of mutually orthogonal functions from μ,with $g_i^n \leq f_n (i = 1,2,...,k_n)$ such that

$$\sum_{i=1}^{k_n} | m(g_i^n) | + \frac{\varepsilon_0}{2} > | m |_0 (f_n)$$

and, hence from (1),

$$\sum_{i=1}^{k_n} | m(g_i^n) | > \frac{\varepsilon_0}{2} \ (n=1,2,...).$$

Since $\{g_i^1\}_{1\le i\le k_1}, \{g_i^2\}_{1\le i\le k_2}, ...\}$ consists of family of mutually orthogonal functions from μ,we have by definition of $|m|_0 (1)$ (for any $\mathrm{N}\ge 1$),

$$|m|_0 (1) \ge \sum_{n=1}^{N}\sum_{i=1}^{k_n} | m(g_i^n) | > \frac{N^{'}\varepsilon_0}{2} ,$$

which contradicts our hypothesis. Hence $|m|_0$ is exhaustive.

For the second part of the proposition, let $\{f_n\}$ be an arbitrary sequence of mutually orthogonal functions from μ, then by Theorem 4.1(i) (iii) and exhaustively of $|m|_0$, we have the assertion.

Propisition4.9. Let m be an real valued function defined on L,with $m(0) = 0$. Then the positive and negative chain variation of m, m_c^+ and m_c^- have the following properties for $f \in \mu$:

(i) $0 \le m_c^+(f) \le \infty, 0 \le m_c^-(f) \ge \infty;$

(ii) $m_c^+(0) = m_c^-(0) = 0;$

(iii) m_c^+ and m_c^- are monotone ;

(iv) $m_c^+ = (-m)_c^-, m_c^- = (-m)_c^+;$

(v) $|m|_c (f) = 0$ if and only if $m_c^+(f) = m_c^-(f) = 0;$

(vi) $m_c^+(f) \ge m_c(f) \ge m_c^-(f);$

Moreover, if m is of bounded chain variation defined on μ, then

(vii) $m = m_c^+ - m_c^-;$

(viii) $|m| = m_c^+ + m_c^-;$

(ix) $m_c^+ = \frac{1}{2}(|m|+m), m_c^- = \frac{1}{2}(|m|-m).$

Proof. Parts (i) - (vii) are the immediate consequence of the Definition 5.1.

(viii) For $f \in \mu$, let us consider the chain $0 = f_0 \le f_1 \le ... \le f_n = f, f_i \in \mu, i = 1,2,...,n.$ Since $| m(f_i) - m(f_{i-1}) | = \max\{m(f_i) - m(f_{i-1}),0\} + \max\{-m(f_i) - m(f_{i-1}),0\}$,we get

$$|m|(f) \le m_c^+(f) + m_c^-(f). \quad (1)$$

For the reverse inequality, for an arbitrary $\varepsilon > 0$, let us choose a chain $0 = f_0 \le f_1 \le ... \le f_n = f, f_i \in \mu, i = 1,2,...,n$ such that

$$\sum_{i=1}^{n} \max\{m(f_i) - m(f_{i-1}),0\} > m_c^+(f) - \varepsilon. \quad (2)$$

Let S_1 be a subchain which consist of chain pairs $\{a_{i-1}, a_i\}$ for which

$$\sum_{S_1} | m(f_i) - m(f_{i-1}) | > m_c^+ - \varepsilon \quad (3)$$

holds, where $\sum_{S_1}$ denotes the summation over the subchain S_1 .Further, let S_2 denotes the subchain which consist of chain pairs $\{f_{i-1}, f_i\}$ of the rest with respect to S_1 .Then we have

$$\sum_{S_1} | m(f_i) - m(f_{i-1}) | - \sum_{S_2} | m(f_i) - m(f_{i-1}) = m(f). \quad (4)$$

Hence $$\sum_{S_{21}} | m(f_i) - m(f_{i-1}) | = \sum_{S_1} | m(f_i) - m(f_{i-1}) - m(f).$$

$$> m_c^+(f) - m(f) - \varepsilon = m_c^-(f) - \varepsilon. \quad (5)$$

Futher, since

$$\sum_{i=1}^{n} | m(f_i) - m(f_{i-1}) = \sum_{S_1} | m(f_i) - m(f_{i-1}) | + \sum_{S_2} | m(f_i) - m(f_{i-1}) = m(f),$$

by use of inequalities (2) and (5), we get

$$\sum_{i=1}^{n} | m(f_i) - m(f_{i-1}) > m_c^+(f) + m_c^- - 2\varepsilon. \quad (6)$$

Since $\varepsilon > 0$ is arbitrary, by inequalities (1) and (6), we have the desired conclusion.

(ix) The result follows from (vii) and (viii).

Theorem4.6. Let m be a function of bounded total variation defined on μ. If there exist monotone real-valued functions γ_1 and γ_2 defined on L, with $\gamma_1(0) = \gamma_2(0) = 0$, such that $m = \gamma_1 - \gamma_2$, then $\gamma_1 \geq m^+$ and $\gamma_2 \geq m^-$, where m_c^+ and m_o^- are the positive and negative chain variations of m, respectively. Moreover, th $\gamma_2 = m_c^- + \gamma$ ere exists a non-negative function γ defined on μ, with $\gamma(0) = 0$, such that $\gamma_1 = m_c^+ + \gamma$ and .

Proof. By proposition 4.9(vii), we get $m_c^+ - m_c^- = m = \gamma_1 - \gamma_2$ and then $\gamma_1 - m_c^+ = \gamma_2 - m_c^-$. To prove the first part of the theorem, it is sufficient to prove that $\gamma_1 - m^+ \geq 0$.

For $f \in \mu$ and arbitrary $\varepsilon > 0$, there exist a chain $0 = f_0 \leq f_1 \leq \ldots \leq f_n = f$, $f_i \in \mu, i = 1,2,\ldots,n$ such that

$$m_c^+(f) - \varepsilon < \sum_{i=1}^{n} \max\{m(f_i) - m(f_{i-1}), 0\},$$

$$= \sum_{i'} [m(f_{i'}) - m(f_{i'-1})],\ m(f_i) - m(f_{i-1}) \geq 0$$

where $\sum_{i'}$ is taken over those i for which. Now, using $m = \gamma_1 - \gamma_2$ and monotonicity of γ_1 and γ_2, we have

$$\sum_{i'} [m(f_{i'}) - m(f_{i'-1})] = \sum_{i'} [\gamma_1(f_{i'}) - \gamma_1(f_{i'-1})] - \sum_{i'} [\gamma_2(f_{i'}) - \gamma_2(f_{i'-1})]$$

$$\leq \sum_{i'} [\gamma_1(f_{i'}) - \gamma_1(f_{i'-1})]$$

$$\leq \sum_{i=1}^{n} [\gamma_1(f_i) - \gamma_1(f_{i-1})] = \gamma_1(f),$$

which yields that $m_c^+(f) - \varepsilon < \gamma_1(f)$. Since $\varepsilon > 0$ is arbitrary, we have desired conclusion.

Putting $\gamma = \gamma_1 - m_c^+ = \gamma_2 - m_c^-$, the second part of the theorem is proved.

Lemma4.1.Let $f, g \in \mu$ and $g \leq f$ .If $| m |_c (g) < \infty$,then $| m(f) - m(g) \leq | m |_c (f) - | m |_c (g)$.

Proof. For an arbitrary $\varepsilon > 0$, there exist a chain connecting 0 to g, that is, $0 = f_0 \leq f_1 \leq \ldots \leq f_n = g$, $f_i \in \mu, i = 1,2,\ldots,n$ such that

$$| m |_c (g) - \varepsilon < \sum_{i=1}^{n} | m(f_i) - m(f_{i-1}) |.$$

Now, let us consider the chain $0 = f_0 \leq f_1 \leq \ldots \leq f_n = g \leq f_{n+1} = f$, $f_i \in \mu, i = 1,2,\ldots,n$. For this special chain, we have the preceeding inequalities as

$$\sum_{i=1}^{n+1} | m(f_i) - (f_{i-1}) | -(| m |_c (g) - \varepsilon) > | m(f) - m(g) | .$$

The definition of $| m | (f)$ yields that

$$| m |_c (f) > (| m |_c (g) - \varepsilon) + | m(f) - m(g),$$

which further yields that

$$| m |_c (f) - m |_c (g) + \varepsilon > | m(f) - m(g) | .$$

Since $\varepsilon > 0$ is arbitrary, we have the desired inequalities.

Theorem 4.7. A real valued function m defined on μ, with $m(0) = 0$ is of bounded chain variation if and only if there exists a monotone real-valued function γ defined on μ such that $| m(f) - m(g) | \le \gamma(f) - \gamma(g)$ holds for any $f, g \in \mu$ and $g \le f$.

Proof. Firstly, let us suppose that m is of bounded chain variation. Putting $\gamma = | m |_c$, we get γ is monotone and by lemma 4.1, we get for $f, g \in \mu$ and $g \le f$, $| m(f) - m(g) | \le \gamma(f) - \gamma(g)$.

Conversely, if γ is monotone real-valued function defined on μ, which satisfy the given inequalities, then for an arbitrary but fixed chain $0 = f_0 \le f_1 \le \ldots \le f_n = 1,, f_i \in \mu, i = 1,2,\ldots,n$, we have

$$\sum_{i=1}^{n} | m(f_i) - m(f_{i-1}) | \le \sum_{i=1}^{n} (\gamma(f_i) - \gamma(f_{i-1})) = \gamma(1) - \gamma(0).$$

Thus, by definition of $| m |_c (1)$ we get

$$| m |_c (1) \le \gamma(1) - \gamma(0).$$

Hence m is of bounded chain variation.

Proposition4.10. Let m be a function of bounded chain variation defined on μ. If its chain variation $| m |_c$ is continuous from below (respectively, continuous from above), then so is m.

Proof. Let $\{ f_n \}$ be an increasing sequence of function from μ. By Lemma 4.1, we have the inequality:

$$| m(\vee_{n=1}^{\infty} f_n) - m(f_n) | \le | m |_c (\vee_{n=1}^{\infty} f_n) - | m |_c (f_n).$$

Thus continuity from below of $| m |_c$ implies that of m.

Continuity from above of m can be proved by similar arguments.

Proposition 4.11 Let m be a function of bounded chain variation defined on μ. Then

(i) m is continuous from below if and only if its chain variation $| m |_c$ iscontinuous from below.

(ii) m is continuous from below if and only if its positive and negative variation m_c^+ and m_c^- are continuous from below.

Proof. The if part: The proof of this part is done in proposition 4.10.

The only if part: Let $\{ f_n \}$ be an increasing sequence of function in μ.

Let us consider an arbitrary finite family $\{g_i\}_{1 \le i \le k}$ of function from μ. Consider the finite chain $0 = f_0 \le f_1 \le \ldots \le f_k = \vee_{i=1}^{\infty} f_n$. Since $(g_i \wedge f_n) \uparrow g_i$ as $n \to \infty$ for each fixed $i (1 \le i \le k)$ and m is continuous from below, we get

$$\sum_{i=1}^{k} | m(g_i) - m(g_{i-1}) | = \sum_{i=1}^{k} | \lim_{n\to\infty} m(g_i \wedge f_n) - \lim_{n\to\infty} (g_{i-1} \wedge f_n) |$$

$$= \sum_{i=1}^{k} \lim_{n\to\infty} | m(g_i \wedge f_n) - m(g_{i-1} \wedge f_n) |$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} | m(g_i \wedge f_n) - m(g_{i-1} \wedge f_n) | .$$

Since the chain $0 = g_0 \wedge f_n \le g_1 \wedge f_n \le \ldots g_k \wedge f_n = f_n$, is a chain connecting 0 to f_n, by definition of $| m | (f_n)$ we get

$$\sum_{i=1}^{k} | m(g_i) - m(g_{i-1}) | \le \lim_{n\to\infty} | m |_c (f_n).$$

Further, since the family $\{g_i\}_{1\le i\le k}$ is arbitrary, by last inequality we get,

$$| m |_c (\vee_{n=1}^{\infty} f_n) \le \lim_{n\to\infty} | m |_c (f_n).$$

Since the opposite inequality follows from the monotonicity of $| m |_c$, we obtain the desired result.
The result is immediate consequence of (i) and Proposition 4.9(7) and (8).

Proposition 4.12. Let m be a real-valued function defined on μ, with $m(0) = 0$.

(i) If m is null-null-additive, then its chain variation $| m |_c$ is also null-null-additive.

(ii) If m is non-negative and monotone, then its chain variation $| m |_c$ is also null-additive.

Proof.(i) The proof this part is similar to Proposition 4.6.

(ii) Let $f, g \in \mu, f \perp g$ and $|m|_c(g) = 0$. Then for any $h \in \mu$, we have by monotonocity of $| m |_c$,

$$| m(h \wedge g) | \le | m |_c (h \wedge g) \le | m |_c (g) = 0,$$

that is, $m(h \wedge g) = 0$. For an arbitrary chain in μ between 0 and $f \vee g : 0 = h_0 \le h_1 \le \ldots \le h_n = f \vee g, h_i \in \mu, i = 1,2,\ldots,n$, we have by monotonocity of $| m |_c$ and null-additivity of m,

$$\sum_{i=1}^{n} | m(h_i) - m(h_{i-1}) | = \sum_{i=1}^{n} | m(h_i \wedge (f \vee g)) - m(h_{i-1} \wedge (f \vee g)) |$$

$$= \sum_{i=1}^{n} | m((h_i \wedge f) \vee (h_i \wedge g)) - m((h_{i-1} \wedge f) \vee (h_{i-1} \wedge g)) |$$

$$= \sum_{i=1}^{n} | m(h_i \wedge f) - m(h_{i-1} \wedge f) |.$$

Further, since $\{h_i \wedge f\}_{1\le i\le n}$ is a special chain between 0 and f, by the definition of $| m |_c (f)$, we have

$$\sum_{i=1}^{n} | m(h_i) - m(h_{i-1}) | \le | m |_c (f).$$

Thus, by the definition of $| m |_c (f \vee g)$, we get

$$| m |_c (f \vee g) \le | m |_c (f).$$

Since monotonocity of $| m |_c$ implies the reverse inequality, we conclude that $| m |_c$ is null-additive.

Theorem 4.8. Let m be an extended real-valued function defined on μ, with $m(0) = 0$. Then is of bounded chain variation if and only if it can be expressed as a difference of two finite, non-negative, monotone real-valued function m_1 and m_2 on μ, with $m_1(0) = m_2(0) = 0$.

Proof. Firstly, let us suppose that m is of bounded chain variation. Then by Proposition 4.9(vii), we have $m = m^+ - m^-$. Put $m_1 = m^+$ and $m_2 = m^-$. Again according to proposition 4.9 (i)-(iii), we obtain that both m_1 and m_2 are finite, non-negative, monotone real-valued function on μ, with $m_1(0) = m_2(0) = 0$.

Conversely, let $m = m_2 - m_2$,where m_1 and m_2 are finite, non-negative, monotone real-valued function on µ, with $m_1(0) = m_2(0) = 0$.Then for $f, g \in \mu, g \le f$,we have by monotonicity of m_1 and m_2,

$$\begin{aligned} | m(f) - m(g) &= | m_1(f) - m_2(f) - (m_1(g) - m_2(g)) | \\ &= | m_1(f) - m_1(g) - (m_2(f) - m_2(g)) | \\ &\le m_1(f) - m_1(g) + m_2(f) - m_2(g) \\ &= m_1(f) + m_2(f) - (m_1(g) - m_2(g)) \\ &= \gamma(f) - \gamma(g), \end{aligned}$$

where $\gamma = m_1 + m_2$. Hence by Theorem 4.7, *m* is bounded chain variation.

Theorem 4.9. (Jordan type decomposition Theorem) . Let *m* be an extended real-valued function defined on μ, with $m(0) = 0$. Then m is of bounded chain variation if and only if it can be expressed as a difference of two finite, non-negative, monotone real-valued function m_1 and m_2 on *L*, with $m_1(0) = m_2(0) = 0$ .Moreover if m is continuous from below function defined on μ, then its decomposed parts m_1 and m_2 are also continuous from below function on μ.

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