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n-DIAMETER AND GEOMETRIC STRUCTURES IN GENERALIZED n-METRIC SPACES WITH AN APPLICATION

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ABSTRACT

We investigate the geometric structure of generalized n-metric spaces through the introduction of n-diameter D_n. We establish a fundamental inequality for the diameter of the union of finite subsets and prove that the n-diameter is invariant under the closure operation in the induced topology. Finally, we propose an application in data science by defining an n-stable clustering criterion, providing a theoretical foundation for multi-point proximity analysis in high-dimensional datasets.

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1. Introduction

The concept of 2-metric space was introduced by Gähler ([3], [4]) as a possible generalization of the usual notion of a metric space. B. C. Dhage [1] introduced the concept of D-metric in order to translate results from usual metric space to D-metric space. The study of generalized metric structures has evolved significantly since the introduction of G-metrics by Mustafa and Sims [6] and subsequent n-metrics. In [5], the author introduced the concept of the generalized n-metric space (G_n-metric space), which provides a robust framework for measuring the distance between n elements of a set simultaneously.

In classical metric spaces, the diameter of a set is a fundamental tool for understanding boundedness and convergence. In G_n-metric spaces, the diameter must account for the interaction of n-tuples. In this paper, we define n-diameter D_n for G_n-metric spaces. We prove a union inequality that generalizes the classical triangle-based diameter bounds and demonstrate that the G_n-topology behaves predictably with respect to the D_n. We formulate a clustering stability theorem applicable to n-ary data relations.

2. Preliminaries

We recall the fundamental definition of a generalized n-metric space as introduced in [5].

Definition 2.1 [5]: Let X be a non-empty set, and $\mathbb{R}^+$ denote the set of non-negative real numbers. Let $G_n: X^n \to \mathbb{R}^+$, $(n \geq 3)$be a function satisfying the following properties:

[G 1]: $G_n(x_1, x_2, \dots \dots . x_n) = 0$ if $x_1 = x_2 = \cdots = x_n$

[G 2]: $G_n(x_1, x_1, \dots \dots . x_1, x_2) > 0$ for all $x_1, x_2 \in X$ with$x_1 \neq x_2$

[G 3]: $G_n(x_1, x_1, \dots \dots . x_1, x_2) \leq G_n(x_1, x_2, \dots \dots . x_n)$ for all $x_1, x_2, \dots \dots, x_n \in X$ with the condition that any two of the points $x_2, \dots \dots, x_n$ are distinct.

[G 4]: $G_n(x_1, x_2, \dots \dots . x_n) = G_n\left(x_{\pi(1)}, x_{\pi(2)}, \dots \dots . x_{\pi(n)}\right)$

for all $x_1, x_2, \dots \dots, x_n \in X$ and every permutation π of $\{1,2, \dots ., n\}$

[G 5]: $G_n(x_1, x_2, \dots \dots ., x_n) \leq G_n(x_1, x_{n+1}, \dots \dots . x_{n+1}) + G_n(x_{n+1}, x_2, \dots \dots ., x_n)$

for all $x_1, x_2, \dots \dots ., x_n, x_{n+1} \in X$

then the function G_n is called a ***Generalized n-metric*** on X, and the pair (X, G_n) a ***Generalized n-metric space***.

Example 2.1.1: Let $\mathbb{R}$ denote the set of all real numbers. Define a function $\rho: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \dots \times \mathbb{R} \to \mathbb{R}$ by

$$\rho(x_1, \dots, x_n) = max\{|x_r - x_s|: r, s \in \{1,2, \dots n\}, r \neq s\}$$

for all $x_1, x_2, \dots, x_n \in X$. Then$(\mathbb{R}, \rho)$ is a generalized n-metric space.

Example 2.1.2: For any metric space(X, d), the following metrics define generalized n-metrics on X:

(1) $G_1^d(x_1, x_2, \dots, x_n) = \sum_r \sum_s d\left(x_r, x_s\right)$

(2)$G_2^d(x_1, \dots, x_n) = max\left\{d\left(x_r, x_s\right): r, s \in \{1,2, \dots n\}, r \neq s\right\}$

Definition 2.2 [5]: Let (X, G_n) be a generalized n-metric space then for $x \in X, r > 0$, the G_n*-ball* with centre x and radius r is

$$B_G(x, r) = \{y \in X: G_n(x, y, y, \dots ., y) < r\}$$

The G_n-metric induces a topology $\tau(G_n)$ where the open balls are defined as $B_G(x, r)$.

3. Main Results

We begin by defining the diameter of a set in this generalized context.

Definition 3.1 (The $\boldsymbol{n}$-Diameter) Let (X, G_n) be a G_n-metric space and $A \subseteq X$ be a non-empty subset of X. The n-diameter of A is defined as

$$D_n(A) = \sup\{G_n(a_1, a_2, \dots \dots . a_n) : a_i \in A\}$$

A set A is said to be G_n-bounded if $D_n(A) < \infty$.

Proposition 3.1 Let $A \subseteq B \subseteq X$. Then $D_n(A) \leq D_n(B)$.

Proof The result follows immediately from the definition of the supremum over a subset.

We now present our first main result regarding the topological closure.

Theorem 3.1 Let (X, G_n) be a G_n-metric space. For any $A \subseteq X$, $D_n(A) = D_n\big(cl(A)\big)$, where $cl(A)$ denotes the closure of A in the topology $\tau(G_n)$.

Proof Since $A \subseteq cl(A)$, we have $D_n(A) \leq D_n\big(cl(A)\big)$ by Proposition 3.1. To show the reverse inequality, let $x_1, \dots, x_n \in cl(A)$. By the definition of closure, for any $\varepsilon > 0$ and for each $i \in \{1,2, \dots, n\}$ there exists $a_i \in A$, such that

$$G_n(x_i, a_i, \dots \dots . a_i) < \frac{\varepsilon}{n-1} \quad \dots \dots \dots (1)$$

By [G 5] we have

$$G_n(x_1, x_2, \dots \dots ., x_n) \leq G_n(x_1, a_1, \dots \dots . a_1) + G_n(a_1, x_2, \dots \dots ., x_n)$$
$$< \frac{\varepsilon}{n-1} + G_n(a_1, x_2, \dots \dots ., x_n)$$

By successive applications of axioms [G 4] and [G 5], replacing each x_i by a_i one at a time, using (1) and $G_n(a_n, a_1, \dots \dots . a_{n-1}) \leq D_n(A)$, we get

$$G_n(x_1, x_2, \dots \dots ., x_n) < (n-1).\frac{\varepsilon}{n-1} + D_n(A) = \varepsilon + D_n(A)$$

Since $\varepsilon > 0$ is arbitrary, hence $G_n(x_1, x_2, \dots \dots ., x_n) \leq D_n(A)$. Taking the supremum, we have $D_n\big(cl(A)\big) \leq D_n(A)$, as $x_1, \dots, x_n \in cl(A)$.■

A critical question in geometry is how the diameter of a union relates to the individual diameters. In G_n-metric spaces, the "cross-distance" must be accounted for.

Definition 3.2 (The $\boldsymbol{n}$-interaction or cross-distance) Let (X, G_n) be a G_n-metric space and $A, B \subseteq X$. The n-interaction (or cross-distance) between A and B is

$$\Delta_n(A, B) = \sup\{G_n(x_1, \dots \dots ., x_n) : \{x_1, \dots \dots ., x_n\} \cap A \neq \emptyset \ and\ \{x_1, \dots \dots ., x_n\} \cap B \neq \emptyset\}$$

Theorem 3.2 (Union equality) Let A and B be non-empty bounded subsets of a G_n-metric space (X, G_n). Then

$$D_n(A \cup B) = \max\{D_n(A), D_n(B), \Delta_n(A, B)\}$$

Proof: Let $x_1, \ldots, x_n \in A \cup B$. Then $x_1, \ldots, x_n \in A$ or $x_1, \ldots, x_n \in B$

If $x_1, \ldots, x_n \in A$, then $G_n(x_1, \ldots, x_n) \leq D_n(A)$.

If $x_1, \ldots, x_n \in B$, then $G_n(x_1, \ldots, x_n) \leq D_n(B)$

If some $x_i \in A \backslash B$ and some $x_j \in B \backslash A$, then by definition 3.2, we have

$$G_n(x_1, \ldots, x_n) \leq \Delta_n(A, B)$$

In any case we have

$$G_n(x_1, \ldots, x_n) \leq \max\{D_n(A), D_n(B), \Delta_n(A, B)\}$$

Taking the supremum over all n-tuples in $A \cup B$, we have

$$D_n(A \cup B) \leq \max\{D_n(A), D_n(B), \Delta_n(A, B)\}$$

To prove the reverse inequality, we note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Therefore, by proposition 3.1, we have $D_n(A) \leq D_n(A \cup B)$ and $D_n(B) \leq D_n(A \cup B)$.

Also every n-tuple that meets both A and B has all its entries in $A \cup B$, so $G_n(x_1, \ldots, x_n) \leq D_n(A \cup B)$. Taking the supremum gives $\Delta_n(A, B) \leq D_n(A \cup B)$. Hence we have

$$\max\{D_n(A), D_n(B), \Delta_n(A, B)\} \leq D_n(A \cup B)$$

Combining both inequalities we get the result. ■

In data analysis, a cluster is often considered stable if the internal distances are significantly smaller than the distances between clusters. We formulate this notion in the G_n-metric setting.

Definition 3.3 Let (X, G_n) be a G_n-metric space. A partition $\mathcal{P} = \{A_1, A_2, \ldots, A_m\}$ of X into non-empty G_n-bounded sets is said to be (α, n)-stable if for every $i \neq j$,

$$\Delta_n(A_i, A_j) \geq \alpha. \max\{D_n(A_i), D_n(A_j)\},$$

where $\alpha > 1$ is the stability coefficient.

Theorem 3.3 Let (X, G_n) be a G_n-metric space and let $\mathcal{P} = \{A_1, A_2, \ldots, A_m\}$ be an (α, n)-stable partition of X with stability coefficient $\alpha > 1$. Let $f: X \to X$ be a mapping satisfying the G_n-contraction condition

$$G_n(f(x_1), f(x_2), \ldots \ldots f(x_n)) \leq kG_n(x_1, x_2, \ldots \ldots x_n) \quad \text{for all } x_1, \ldots, x_n \in X \quad \ldots.(2)$$

with constant $k \in (0,1)$. Suppose $f(A_i) \subseteq A_i$ for every $i \in \{1, \dots, m\}$. Then

(a) $D_n\big(f(A_i)\big) \le kD_n(A_i)$ for every i;

(b) $\Delta_n\Big(f(A_i), f\big(A_j\big)\Big) \le k\Delta_n\big(A_i, A_j\big)$ for every $i \ne j$;

(c) If equality holds in the contraction condition (2), then the image $f(\mathcal{P}) = \{f(A_1), f(A_2), \dots, f(A_m)\}$ is (α, n)-stable with the same stability coefficient α.

Proof (a) For any $x_1, x_2, \dots\dots. x_n \in A_i$, the contraction condition (2) and the definition of D_n give

$$G_n\big(f(x_1), f(x_2), \dots\dots. f(x_n)\big) \le kG_n(x_1, x_2, \dots\dots. x_n) \le kD_n(A_i)$$

Taking supremum over all n-tuples from A_i and using $f(A_i) \subseteq A_i$, we have

$$D_n\big(f(A_i)\big) = \sup G_n\big(f(x_1), f(x_2), \dots\dots. f(x_n)\big) \le kD_n(A_i)$$

(b) Let $(y_1, \dots. y_n)$ be any n-tuple with $\{y_1, \dots. y_n\} \cap f(A_i) \ne \emptyset$ and $\{y_1, \dots. y_n\} \cap f\big(A_j\big) \ne \emptyset$. Since $f(A_i) \subseteq A_i$ and $f\big(A_j\big) \subseteq A_j$, we can write $y_l = f(x_l)$ for each l, where $x_l \in A_i \cup A_j$ and the corresponding tuple meets both A_i and A_j. By (2), we have

$$G_n(y_1, \dots, y_n) = G_n\big(f(x_1), \dots, f(x_n)\big) \le kG_n(x_1, \dots, x_n) \le k\Delta_n\big(A_i, A_j\big)$$

Taking the supremum over all mixed n-tuples in $f(A_i) \cup f\big(A_j\big)$, we have

$$\Delta_n\Big(f(A_i), f\big(A_j\big)\Big) \le k\Delta_n\big(A_i, A_j\big)$$

(c) Suppose the equality holds in the contraction condition (2). Then from (a) and (b), we have

$$D_n\big(f(A_i)\big) = kD_n(A_i) \text{ and } \Delta_n\Big(f(A_i), f\big(A_j\big)\Big) = k\Delta_n\big(A_i, A_j\big) \text{ for every } i \ne j$$

Therefore,

$$\frac{\Delta_n\Big(f(A_i), f\big(A_j\big)\Big)}{\max\Big\{D_n\big(f(A_i)\big), D_n\Big(f\big(A_j\big)\Big)\Big\}} = \frac{k\Delta_n\big(A_i, A_j\big)}{k\max\big\{D_n(A_i), D_n\big(A_j\big)\big\}} = \frac{\Delta_n\big(A_i, A_j\big)}{\max\big\{D_n(A_i), D_n\big(A_j\big)\big\}} \ge \alpha$$

Hence $f(\mathcal{P})$is (α, n)-stable with the same stability coefficient α.■

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