ANALYSIS ON STABILITY OF PERIODIC POINTS, PERIOD DOUBLING BIFURCATION AND LYPUNOV EXPONENT IN A CHAOTIC MODEL

[1]Tarini Kumar Dutta & [2]Pramila Kumari Prajapati

Department of Mathematics, Gauhati University,Guwahati-781014, India.

ABSTRACT

In this paper we consider a non linear algebraic mathematical model $f(x) = \mu x e^{\frac{-c}{v}x}$ *with μ as an adjustable parameter which is modified form of "Ricker Population Model" . With the help of programming in MATHEMATICA it has been observed that the map follows a period doubling route to chaos. Further in order to establish a universal route from order to chaos through period doubling bifurcations we built up appropriate numerical methods to obtain periodic points and bifurcation points of different periods* $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$*..and find Feigenbaum Universal Constant* (δ) *=4.66920318….. Again with the help of bifurcation points and Feigenbaum delta, the accumulation point is calculated numerically. We confirm the chaotic nature of the model through Schwarzian derivatives which is found to be negative, and chaotic region has also been confirmed by obtaining positive Lyapunov Exponents at suitable parametric values.*

Key words: Feigenbaum Universality / Chaos/Schwarzian Derivative/ Lyapunov Exponent

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1. Introduction

Here we discuss a detailed analysis of period doubling bifurcation of the model, we study some associated universalities, particularly the route from order to chaos, as developed by Mitchell J. Feigenbaum, an American Physicist. M. J. Feigenbaum discussed the universal behaviors of one dimensional unimodel map of the form $x_{n+1} = \mu f(x_n)$. In this paper we consider a one dimensional non linear algebraic mathematical model $f(x) = \mu x e^{-x/300}$ where $x \in [0,300]$ and $\mu \in [7,20]$ is an adjustable parameter .Chaos and order have long been viewed as antagonistic in the science. One of the great surprises revealed through the studies of the cubic iterator $x_{n+1} = \mu x_n e^{-x_n/300}$, n = 0,1, 2, 3……..is the discovery that there is a very well defined period-doubling route to chaos[11].

We now introduce some basic concepts, which we are using in this paper.

1.1 Fixed Point

Let $f: X \to X$ be a differentiable map where X is an interval on the real line. A point $x^* \in X$ is called a fixed point of X if $f(x^*) = x^*$ [1,5].

In this paper, our mathematical model is $f(x) = \mu x e^{-x/300}$ where $x \in [0,300]$and $\mu \in [7,20]$.Clearly solution of $f(x)=x$ gives the fixed points of f. A fixed point $x^*$is said to be (i) Stable fixed point or attractor if $|f'(x)|<1$, (ii) Unstable fixed point or repeller if $|f'(x)|>1$,(iii) Super attractive or super stable if $f'(x)=0$.

1.2 Periodic point

Let us suppose the notation f^nindicate the n-fold composition i.e.

$$f^n(x) = f.f.f.f \ldots \ldots f(x) \quad (n \text{ times})$$

A point x is called a periodic point of period n if $f^n(x) = x$ when n is a smallest positive integer.

From this definition we can easily say that the fixed points can be termed as periodic point of period one. In the similar manner, a periodic point of period n of a map f can be thought as the fixed point(s) of the n-th iteration of the map [2], [5], [8].

1.2 Bifurcation and Bifurcation Point

The word 'Bifurcation' literally means splitting into two parts. It is often desirable to know how the fixed points of a system change when a parameter of the system is changed. Normally a gradual variation of a parameter in the system corresponds to the gradual variation of the solutions of the problem. However there exists a large number of problems for which the number of solutions changes abruptly and the structure of solution manifolds varies dramatically when a parameter passes through some critical values (fixed values) .This qualitative change in the structural behavior of the system parameter values are called bifurcation points $f'(x) = -1$ indicates bifurcation for unimodal map [2, 4, 8].

1.3 Chaos

Generally there is no accepted definition of Chaos. From practical point of view, chaos can be defined as bounded steady state behavior that is not an equilibrium point, not periodic and not quasi periodic. The trajectories are, indeed bounded. They are not periodic and they do not have periodic distribution characteristics of quasi periodic solutions. A noise like spectrum is a characteristic of chaotic systems. Another important fact about the chaotic systems in that the limit set for chaotic behaviors is not a simple geometrical object like circle or torus, but is related to fractals. [5,6]

In short chaos can be defined as effectively unpredictable long time behavior arising in a deterministic dynamical system because of sensitivity to the initial conditions. The key to long term unpredictability is a property known as sensitivity to (or sensitive dependence on) initial conditions. Examples of such systems include the atmosphere, the solar system, plate tectonics, turbulent fluids, economic and population growth.[4,5]

For Devaney, chaos is seen as mixing of unpredictability and regular behaviors: a system is chaotic in the sense of Devaney if it is transitive, sensitive to the initial conditions and has a dense set of periodic points [2].

2. Study for our model and discussion

Taking above definition in our mind, we try to find the fixed points, nature of the points. Before finding this, we note that *f(x)* is maximum at *x*=300 as $f''(x) < 0$ for $\mu > 1$.

At this point the maximum value for *f(x)* is $300\mu/e$ so if we take $\mu = 7.3891$and *2<e<3* then we have the maximum value of f(x) is x= 738.91. The range may be taken as [0,738.91]. Now solving *f(x)=x* we get *x*=0, x=300 log μ .We have

$$f'(x) = \mu\left(xe^{-\frac{x}{300}}\left(-\frac{1}{300}\right) + e^{-\frac{x}{300}}\right).$$

Now the derivative at x=0 is$|f'(x)|_{x=0} = \mu > 1$,so x=0 is unstable fixed point.

$$|f'(x)|_{x=300\log\mu} = 1 - \log\mu.$$

The absolute value remains less than 1 for 1<μ <7.38910…..; so is x=300 log μ a stable point for $1 < \mu < 7.38910\ldots$ and when $\mu > 7.38910..$ this fixed point becomes unstable. Therefore the first point of bifurcation of our model is 7.38910….

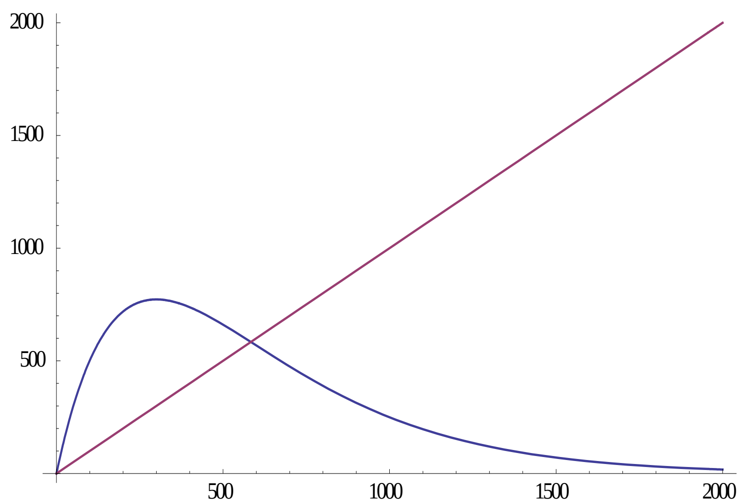

Figure 1.a: Graph of $f(x) = \mu x e^{-x/300}$ and f(x)=x at the parameter $\mu = 7.3891$ ….

Their intersection points give the fixed points of *f*.

Next we consider the iterated map $f^2(x)$. The fixed points of f^2 are given by solving the equation $f^2(x) = x$. The parameter at where these two fixed points start to become unstable is the second bifurcation point. To find bifurcation point we take the help of c programming and Newton Raphson Method. The second bifurcation point is $\mu_2 = 12.52999 \ldots \ldots$…..

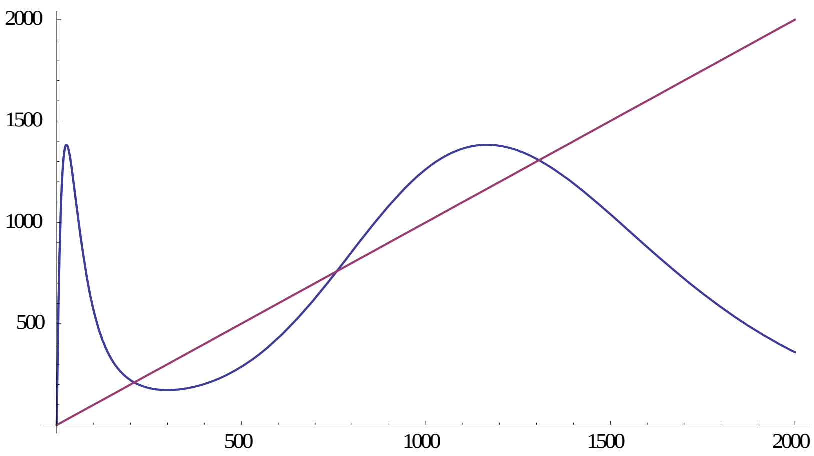

Figure 1.b: Graphs of $f^2(x)$ and *f(x)*=*x* at the parameter $\mu = 12.5299$ …… Their intersection gives four fixed points of $f^2(x)$.

Now we try to find the third bifurcation point. For this we solve the equation $f^4(x) = x$, So to find one of the periodic points we take the help of C-programming and Newton –Rapson method. We see that the periodic point are 198.6072, 1283.6089, 222.9304, 1328.604 and the corresponding bifurcation point is μ_3=14.244249……..

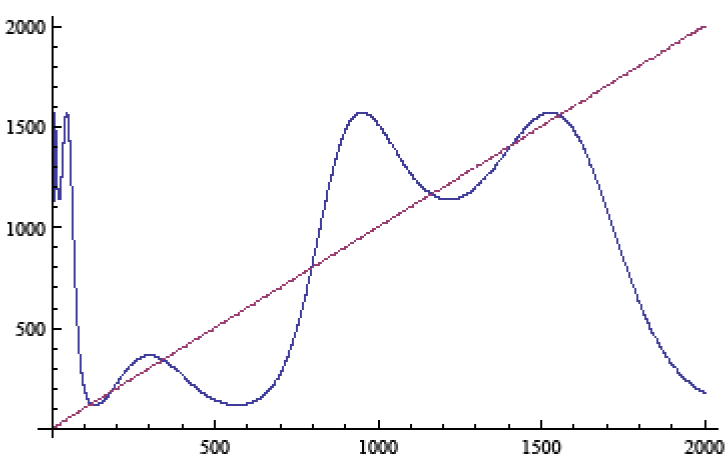

Figure 1.c: Graph of $f^4(x)$ and f(x)= x at the parameter $\mu = 14.244249$ …. Their intersection gives the fixed points of f^4

2.1 Numerical Algorithm to find periodic points, Derivatives of different Iterates of our map and the Bifurcation points.

(i) To find a fixed point or a periodic point of our model, we apply Newton-Recurrence formula:

$$x_{n+1} = x_n - \frac{g(x_n)}{\frac{d}{dx}g(x_n)} \quad \text{where, n=1,2,3,4………}$$

The Newton formula [13] actually gives the zero(es) of a map and to apply this numerical tool, one needs a number of recurrence formulae which are given below:

Let the initial value of x be x_0. Then,

$$f(x_0) = \mu x_0 e^{-x_0/300} = x_1 \quad \text{(say)}$$

$$f^2(x_0) = f(x_1) = \mu x_1 e^{-x_1/300} = x_2 \quad \text{(Say)}$$

Proceeding in this manner, the following recurrence formula can be established

$$x_n = \mu x_{n-1} e^{-x_{n-1}/300} \qquad \text{n=1,2,3,4……..}$$

(ii) Again the derivative of f^k can be obtained as follows

$$|\frac{df}{dx}|_{x=x_0} = \mu[\frac{-1}{300}x_0 e^{\frac{-1}{300}x_0} + e^{\frac{-1}{300}x_0}]$$

Again by chain rule of differentiation we get

$$\left|\frac{df^2}{dx}|_{x=x_0} = \left|\frac{df}{dx}|_{f(x_0)}\right|\frac{df}{dx}|_{x=x_0} = \left(\mu\left[\frac{-1}{300}x_1 e^{\frac{-1}{300}x_1} + e^{\frac{-1}{300}x_1}\right]\right)(\mu[\frac{-1}{300}x_0 e^{\frac{-1}{300}x_0} + e^{\frac{-1}{300}x_0}])\right|$$

Where $x_1 = f(x_0)$.

Proceeding in this way we can obtain

$$|\frac{df^k}{dx}|_{x=x_0} = \left(\mu\left[\frac{-1}{300}x_{k-1}e^{\frac{-1}{300}x_{k-1}} + e^{\frac{-1}{300}x_{k-1}}\right]\right) \dots\dots (\mu[\frac{-1}{300}x_0 e^{\frac{-1}{300}x_0} + e^{\frac{-1}{300}x_0}])$$

We remind that the value will be the bifurcation value for the map f^k when its derivative $\frac{df^k}{dx}$ at a periodic point is equal to -1.

We now give below a table of the bifurcation points, and of the fixed points(periodic points) at the corresponding bifurcation point and experimental Feigenbaum delta value

$$\delta_n = \frac{\mu_{n+1}-\mu_n}{\mu_{n+2}-\mu_{n+1}}$$

Table 2.d(Numerical calculation of bifurcation point)

Bifurcation Point	One of the periodic points	Feigenbaum delta (experimental value)
μ_1 =7.3891078	600.001814	
μ_2=12.529999976599	210.075290	
μ_3=14.244249268501072000	124.320083	δ_1=2.998920678110
μ_4=14.652681529859410900	132.777483	δ_2=4.780306005304
μ_5=14.742121472135032400	126.301161	δ_3=4.566553275489
μ_6=14.761364486925174300	128.252314	δ_4=4.647917348244
μ_7=14.765489896880687600	129.024209	δ_5=4.6645593631323
μ_8=14.766373620799544900	124.205609	δ_6=4.6679547190394
μ_9=14.766562896075338800	124.760090	δ_7=4.6691894913947
μ_{10}=14.766603433429175100	124.527181	δ_8=4.66915715736792
μ_{11}=14.766612115319375700	124.477123	δ_9=4.66918526936483
μ_{12}=14.766613974713006700	124.456834	δ_{10}=4.66920509126173
μ_{13}=14.766614372938601500	124.461724	δ_{11} =4.669196649697
μ_{14} =14.766614458226300100	124.475134	δ_{12}=4.66920323132798

We can see from Table 2.1 that Feigenbaum delta converges to 4.6692031…..Now the bifurcation diagram indicates the universal route to chaos for our model

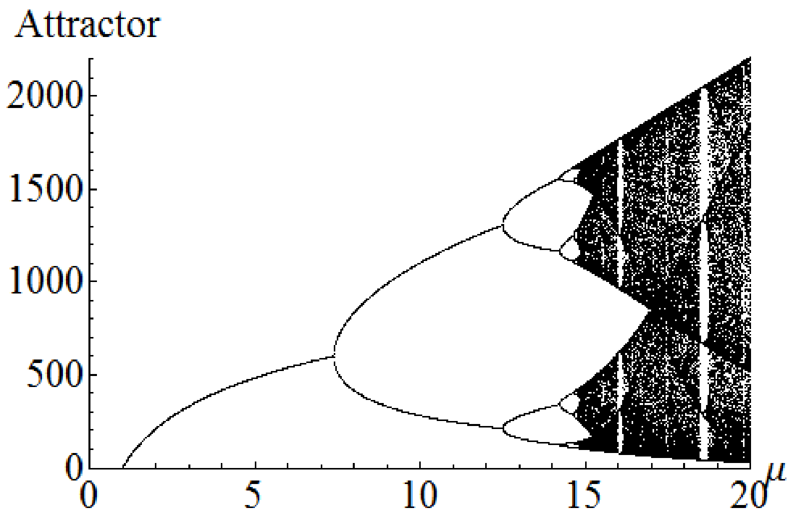

Figure 1.e: Bifurcation diagram of $f(x) = \mu x e^{\frac{-1}{300}x}$

Schwarzian Derivative

The Schwarzian derivative of a function f(x) which is defined in the interval (a,b) having higher order derivatives is given by $S(f(x)) = \frac{f'''[x]}{f''[x]} - \frac{3}{2}[\frac{f''[x]}{f'[x]}]^2$

The derivative was first formulated by H.A. Schwarzian and has been used in the theory of differential equation. It has found important application in the study of periodic orbits, the Schwarzian derivative is used to study the limiting behavior of dynamical systems [9].If f has negative Schwarzian derivative on [0,1], then it turns out that there must be a number c in (0,1) such that $f'(c) = 0$, that is, f has a critical point. We say that f is S- Unimodal if it's Schwarzian derivative is negative [9,10]. A surprise hidden in the above formula is that the Schwarzian Derivative is actually not a function. Informally speaking Schwarzian derivative is a curvature. [10]

A sufficient condition for a function from interval to interval to behave chaotically is that its Schwarzian derivative is negative. [7]

Here we see that after calculating $f'(x), f''(x), f'''(x)$ we get

$\mathrm{S}(\mathrm{f}(\mathrm{x})) = \frac{5472000000 - 42360000\,x + 100600\,x^2 - 67x^3}{20000\,(-600+x)(-300+x)^2} < 0$ for all x.

Accumulation point

As our model follows a period doubling bifurcation and so we let $\{\mu_n\}$ be the sequence of the bifurcation points. Using Feigenbaum δ, if we know the first (μ_1) and second(μ_2) bifurcation points then we can expect third bifurcation point (μ_3) as

$$\mu_3 \approx \frac{\mu_2-\mu_1}{\delta}+\mu_2 \qquad \text{(i)}$$

(Of course occurrence of first two period-doubling does not give assurance that a third will occur, but if it does not occur, then it can be predicted by the above equation). Similarly

$$\mu_4 \approx \frac{\mu_3-\mu_2}{\delta}+\mu_3 \qquad \text{(ii)}$$

So (i) and (ii) implies

$$\mu_4 \approx (\mu_2-\mu_1)\left(\frac{1}{\delta}+\frac{1}{\delta^2}\right)+\mu_2$$

Continuing this procedure to obtain μ_5, μ_6 and so on, we get more terms in the sum involving powers of $\frac{1}{\delta}$ and clearly this sum is familiar with G.P series. We can sum the series to obtain the following result [8]

$$\mu_\infty \approx \frac{(\mu_{n+1}-\mu_n)}{\delta-1}+\mu_{n+1}$$

However this expression is exact when the bifurcation ratio $\delta_n = \frac{\mu_{n+1}-\mu_n}{\mu_{n+2}-\mu_{n+1}}$ is equal for all value of n. In fact $\{\delta_n\}$ converges as $n\rightarrow\infty$, that is $\delta_n = \delta$.

So, we consider the sequence $\{\mu_{\infty,n}\}$, $\mu_{\infty,n} = \frac{\mu_{n+1}-\mu_n}{\delta-1}+\mu_{n+1}$, where μ_n are the experimental points. Obviously $(\lim_{n\rightarrow\infty}\mu_{\infty,n} = \mu_\infty)$.

Using the experimental bifurcation points the sequence of accumulation points $\{\mu_{\infty,n}\}$ is calculated for some values of n, as given below.

$\mu_{\infty,1}$=15.1018339776494 $\mu_{\infty,2}$=14.7804306005304 $\mu_{\infty,3}$=14.7671988887239

$\mu_{\infty,4}$= 14.7666395556708 $\mu_{\infty,5}$=14.7718668549468 $\mu_{\infty,6}$=14.7666158751694

$\mu_{\infty,7}$=14.7666145469893 $\mu_{\infty,8}$=14.7666144844159 $\mu_{\infty,9}$=14.7666144816098

$\mu_{\infty,10}$=14.7666144814761 $\mu_{\infty,11}$=14.7666144814707 $\mu_{\infty,12}$=14.7666144814707

The above sequence converges to the value 14.7666144…. which is the required accumulation point.[4.6]

Calculation of the Lyapunov exponent

A quantitative measure of the sensitive dependence to initial conditions is given by the Lyapunov exponents, which measures the exponential separation of the nearby orbits. In simple terms, a positive Lyapunov exponent can be considered to be an indicator of chaos, whereas negative exponents are associated with regular behavior (periodic orbits).[2]

There is some standard procedure for obtaining the Lyapunov exponent and the procedure is as follows. We begin by considering an attractor point x_0 and calculate the Lyapunov exponent, which is the average of the sum of logarithm of the derivative of the function at the iteration points. With the help of a computer program we have followed the procedure to get the Lyapunov exponent. The formula may be summarized as follows:[1,5,6]

Lyapunov exponent (λ)=

$$(\log|f'(x_0)| + log|f'(x_1)| + log|f'(x_2)| + \log|f'(x_3)| + \quad \ldots\ldots + \log|f'(x_n)|)$$

We see several points (the 1st at $\mu_1 = 7.38910$, 2nd at $\mu_2 = 12.5299999765\ldots$.,3rd at $\mu_3 = 14.24424926\ldots$,) where the Lyapunov exponent hits the horizontal line and the becomes negative. These are the period doubling points. Lyapunov exponents at these points are zero (which is atmost clear from figure). The first chaotic region, appears after the value 14.7666144……..

After the chaotic region we see some portions of the graph below the horizontal line. They signify that within the chaotic region also, at certain values of the parameter, there are regular behaviors and after that again chaotic region starts. These are called periodic windows in the chaotic region.

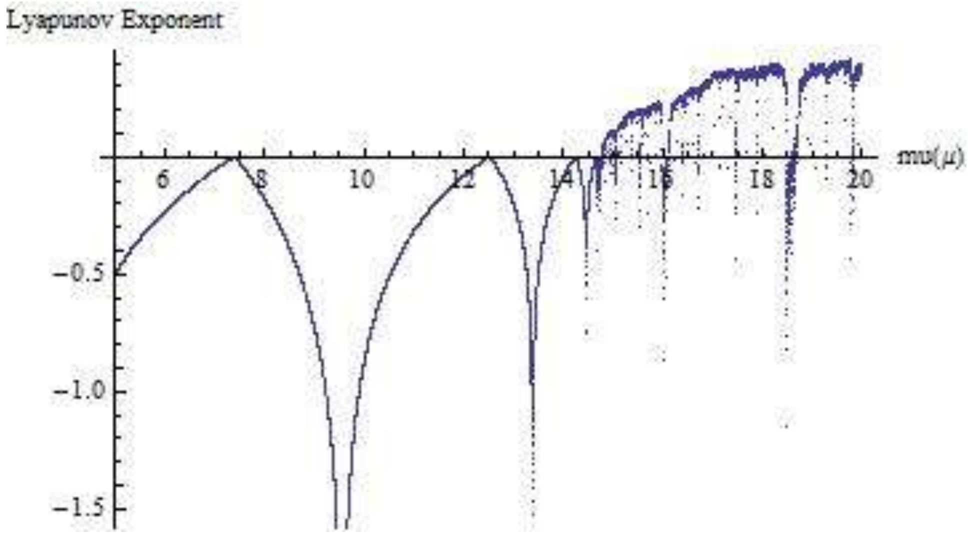

Figure 1.7: graph of Lyapunov Exponent for the parameter from 7.0 to 20.0

6. Result and discussion

In this paper we have seen that simple unimodal mathematical model exhibit periodic doubling route to chaos. Also we have seen that our model successfully converges to Feigenbaum universal constant (delta). From the bifurcation graph we can see that the chaotic region occurs beyond accumulation point 14.766614…as desired. We see from the graph how Lyapunov Exponent changes its sign from negative to positive as the control parameter varies. This negative value indicates about regular behavior of the periodic points and positive values gives the signal of chaos.

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